Self–duality of ASEP with two particle types via symmetry of quantum groups of rank two

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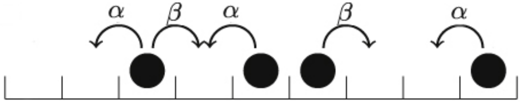
Consider the asymmetric simple exclusion process (ASEP):



Let $q = \sqrt{\beta/\alpha}$ (or $\tau = \beta/\alpha$).
Denote the occupation variables by $\eta_i \in \{0, 1\}$ for each lattice site i.
Denote the particle variables by $\xi^{(n_1,\ldots,n_r)}$.

ASEP has $\mathcal{U}_q(\mathfrak{sl}_2)$ symmetry, which can be used to prove stochastic self–duality.
Recall that $\mathfrak{sl}_2$ is the Lie algebra of traceless 2×2 matrices, with basis

$$e := \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad f := \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad h := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

The universal enveloping algebra $\mathcal{U}(\mathfrak{sl}_2)$ is the algebra generated by e, f, h with the relations

$$ef - fe = h, \quad he - eh = 2e, \quad hf - fh = -2f$$

and co–product $\Delta : \mathcal{U}(\mathfrak{sl}_2) \to \mathcal{U}(\mathfrak{sl}_2) \otimes \mathcal{U}(\mathfrak{sl}_2)$ defined on generators by

$$\Delta(e) = 1 \otimes e + e \otimes 1, \quad \Delta(f) = 1 \otimes f + f \otimes 1, \quad \Delta(h) = 1 \otimes h + h \otimes 1.$$

The quantum group $\mathcal{U}_q(\mathfrak{sl}_2)$ is the algebra generated by $e, f, k = q^h, k^{-1} = q^{-h}$ with the q–deformed relation

$$ef - fe = \frac{q^h - q^{-h}}{q - q^{-1}}$$

and q–deformed co–product

$$\Delta(e) = q^h \otimes e + e \otimes 1, \quad \Delta(f) = 1 \otimes f + f \otimes q^{-h}.$$

In the $q \to 1$ limit, one recovers $\mathcal{U}(\mathfrak{sl}_2)$ from L'Hôpital's rule.

Relationship to ASEP: $\mathfrak{sl}_2$ (and also $\mathcal{U}_q(\mathfrak{sl}_2)$) has a natural two–dimensional representation V with basis $\{v_1, v_2\}$. Using the co–product, $V^{\otimes L}$ is a 2^L–dimensional representation.

Each basis element of $V^{\otimes L}$ corresponds to a state of ASEP on $\{1, \ldots, L\}$, with v_1 corresponding to a particle and v_2 corresponding to an empty site. The elements e and f act as creation and annihilation operators, respectively, and every element of $\mathcal{U}_q(\mathfrak{sl}_2)$ is a symmetry (i.e. commutes with) of the Hamiltonian of ASEP.

This symmetry can be used to prove self–duality. Recall that two Markov processes $X(t), Y(t)$ are dual with respect to $D(\cdot,\cdot)$ if

$$\mathbb{E}_x[D(X(t),y)] = \mathbb{E}_y[D(x,Y(t))].$$

Schutz '95 shows that ASEP is self–dual with duality function

$$D(\eta,\xi^{(n_1,\ldots,n_r)}) = \prod_{s=1}^{r} 1_{\{\eta_{n_s}=1\}} q^{2N_{n_s}(\eta)+2n_s}$$

where $N_i(\eta)$ is the number of particles to the right of site i.

Carinci–Giardinà–Redig–Sasamoto (14) lay out a scheme for generalizing to other Lie algebras $\mathfrak{g}$ and representations V

- Start from $\mathcal{U}_q(\mathfrak{g})$ with a co–product $\Delta : \mathcal{U}_q(\mathfrak{g}) \to \mathcal{U}_q(\mathfrak{g}) \otimes \mathcal{U}_q(\mathfrak{g})$.
- Compute the action of $\Delta(C)$ on $V \otimes V$ for a central element C.
- Construct a quantum Hamiltonian by letting $\Delta(C)$ act on neighboring sites in $V^{\otimes L}$

$$H := \sum_{i=1}^{L} \underbrace{1 \otimes \cdots \otimes 1}_{i-1 \text{ times}} \otimes \Delta(C) \otimes \underbrace{1 \otimes \cdots \otimes 1}_{L-i-1 \text{ times}}$$

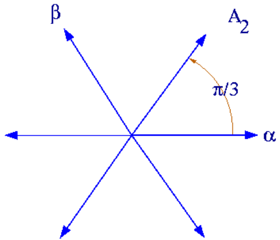
By construction, H has $\mathcal{U}_q(\mathfrak{g})$ symmetry, i.e. $[S, H] = 0$ for every $S \in \mathcal{U}_q(\mathfrak{g})$.

- If H is has non–negative off–diagonal entries, then apply an appropriate conjugation with a diagonal operator G to get a Markov generator $\mathcal{L} = G^{-1}HG$.
- If H is symmetric, then self–duality functions arise of the form $D = G^{-1}SG^{-1}$:

$$\mathcal{L}D = G^{-1}HSG^{-1} = (G^{-1}SG^{-1})(GH^*G^{-1}) = D\mathcal{L}^*$$

Carinci–Giardinà–Redig–Sasamoto (14) do this for $\mathfrak{g} = \mathfrak{sl}_2$ and arbitrary finite–dimensional representations V.

Consider $\mathfrak{gl}_3$, the Lie algebra of 3×3 matrices, with V the natural 3–dimensional representation. Its root system is two–dimensional of type A_2:



Gould–Bracken–Zhang (91) show that

$$q^{-2}k_{(2,0,0)} + k_{(0,2,0)} + q^2 k_{(0,0,2)}$$
$$+(q-q^{-1})^2\Big(q^{-1}k_{(1,1,0)}e_1f_1+qk_{(0,1,1)}e_2f_2+qk_{(1,0,1)}(e_1e_2-q^{-1}e_2e_1)(f_2f_1-q^{-1}f_1f_2)\Big)$$

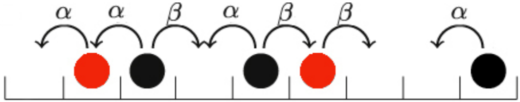
is central in $\mathcal{U}_q(\mathfrak{gl}_3)$.

The proof applies Drinfeld's central element construction to Jimbo's quantum $R(x)$ matrix.

Proposition (K.)

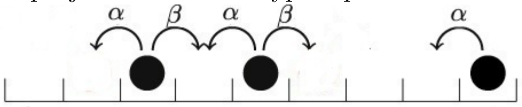
The quantum Hamiltonian H is symmetric with non–negative off–diagonal entries, and there is a G such that $G^{-1}HG$ is the generator of a particle system called spin 1/2 type A_2 ASEP.

Red particles are type 2 (second–class) particles, and type 1 (first–class) particles can switch places with type 2 particles.

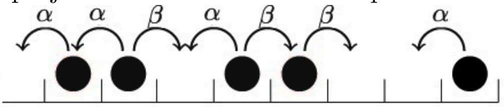


Two interesting projections:

The projection onto the type 1 particles is usual ASEP:



The projection onto the number of particles is also usual ASEP:



Theorem (K.)

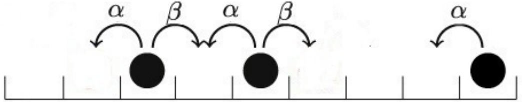
If ξ denotes the state with type 1 particles at $n_1, \ldots, n_r$, and type 2 particles at $m_1, \ldots, m_{r'}$ then

$$D(\eta, \xi) = \prod_{s=1}^{r} 1_{\{\eta_{n_s}=1\}} q^{2\tilde{N}_{n_s}(\eta)+2n_s} \prod_{s'=1}^{r'} 1_{\{\eta_{m_{s'}} \neq 0\}} q^{2N_{m_{s'}}(\eta)+2m_{s'}}$$

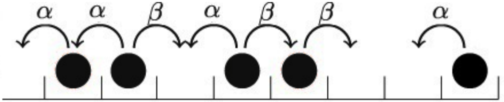
is a self–duality function, where $\tilde{N}_i(\eta)$ is the number of type 1 particles to the right of site i, and $N_i(\eta)$ is the total number of particles to the right of site i.

Two interesting cases:

When ξ consists only of type 1 particles, $D(\eta, \xi)$ is the Schütz duality function for the projection to type 1 particles:



When ξ consists only of type 2 particles, $D(\eta, \xi)$ is the Schütz duality function for the projection to the number of particles:

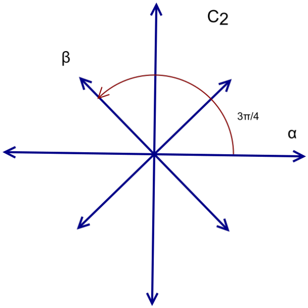


Belitsky–Schütz proved similar results at around the same time, using the Perk–Schultz quantum spin chain to prove $\mathcal{U}_q(\mathfrak{gl}_3)$ symmetry.

Consider $\mathfrak{sp}_4$, the Lie algebra of 4×4 matrices of the form

$$\left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} : A = -D^T, B = B^T, C = C^T \right\}$$

with V the natural four–dimensional representation. Its root system is two–dimensional of type C_2:



Proposition (K.)

The element

$$q^{-4}k_{(-2,0)} + q^{-2}k_{(0,-2)} + q^4 k_{(2,0)} + q^2 k_{(0,2)}$$
$$+ (q-q^{-1})^2(q^{-3}f_1k_{(-1,-1)}e_1 + q^3 f_1 k_{(1,1)}e_1) + (q^2-q^{-2})^2 f_2e_2$$
$$+ (q-q^{-1})^2\Big(q^{-1}(qf_1f_2 - q^{-1}f_2f_1)k_{(-1,1)}(qe_2e_1 - q^{-1}e_1e_2)$$
$$+ q(qf_2f_1 - q^{-1}f_1f_2)k_{(1,-1)}(qe_1e_2 - q^{-1}e_2e_1)$$
$$+ \left(f_1f_1f_2 - (q^{-2}+q^2)f_1f_2f_1 + f_2f_1f_1\right)\left(e_1e_1e_2 - (q^{-2}+q^2)e_1e_2e_1 + e_2e_1e_1\right)\Big)$$

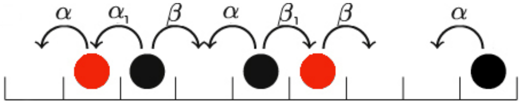
is central in $\mathcal{U}_q(\mathfrak{sp}_4)$.

The proof uses an explicit construction of the quantum Harish–Chandra isomorphism.

In this case, $\Delta(C)$ does not have non–negative off–diagonal entries, which would correspond to a "negative probability" of having a site occupied by both a type 1 and a type 2 particle. However, we can take this probability to be $-\epsilon \to 0$.

Proposition (K.)

The quantum Hamiltonian H is symmetric, and there exist G_ϵ such that $\lim_{\epsilon\to 0} G_\epsilon^{-1} H G_\epsilon$ is the generator of spin 1/2 type C_2 ASEP.



Here $\sqrt{\beta_1/\alpha_1} = q^2$ and $\alpha_1 = q^2(q^2+q^{-2})^2(q^{-4}+q^6)^{-1}\alpha$. The projection to type 1 particles is not Markov.

Let ($\tilde{N}_i^L$) N_i^L be the number of (type 1) particles to the *left* of site i.

Proposition (K.)

Spin 1/2 type C_2 ASEP is self–dual with respect to the function

$$D(\eta,\xi)=\prod_{i=1}^{L}\Big(1_{\{\xi_i=\eta_i=1\}}q^{2(i-1)}+1_{\{\xi_i=\eta_i=2\}}q^{2(i-1+N_i^L(\eta)+N_i^L(\xi))}$$
$$+1_{\{\xi_i=1,\eta_i=2\}}q^{2(N_i^L(\eta)+i-1+2N_i^L(\xi)-\tilde{N}_i^L(\xi))}\Big)$$

and is dual to usual ASEP with respect to the function

$$D(\eta,\xi^{(n_1,\ldots,n_r)})=\prod_{s=1}^{r}1_{\{\eta_{n_s}\neq 0\}}q^{2N_{n_s}(\eta)+2n_s}$$

Thank you!