The two-time distribution in random growth

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Directed last-passage times

Consider the *last-passage times*

$$G(m,n) = \max_{\pi:(1,1)\nearrow(m,n)} \sum_{(i,j)\in\pi} w(i,j),$$

where $w(i,j)$ are i.i.d. geometric random variables

$$\mathbb{P}[w(i,j) = k] = (1-q)q^k, \quad k \geq 0.$$

Directed last-passage times

We have the limit theorem

$$\mathbb{P}\left[\frac{G(n,[\lambda n]) - an}{bn^{1/3}} \leq \xi\right] \to F_2(\xi)$$

as $n \to \infty$, where

$$F_2(\xi) = \det(I - K_{\mathrm{Ai}})_{L^2(\xi,\infty)}$$

is the *Tracy-Widom distribution*, K_{Ai} the *Airy kernel*.

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Fluctuation exponent $1/3$.

Two-point distribution, space direction

Joint distribution of $G(m_1, n_1)$ and $G(m_2, n_2)$ when $m_1 < m_2$ and $n_1 > n_2$.

$$\mathbb{P}\left[\frac{G(n+\nu_1 n^{2/3}, n-\nu_1 n^{2/3}) - an}{bn^{1/3}} \leq \xi_1, \frac{G(n+\nu_2 n^{2/3}, n-\nu_2 n^{2/3}) - an}{bn^{1/3}} \leq \xi_2\right]$$

converges to a Fredholm determinant involving the extended Airy kernel.

Fluctuation exponent $2/3$.

Two-point distribution, time direction

Two-time joint distribution

$$\mathbb{P}\left[G(m_1, m_1) \leq v_1, G(m_2, m_2) \leq v_2\right]$$

We expect non-trivial fluctuations if

$$\frac{m_1}{m_2 - m_1} \sim c, \quad c > 0.$$

Exponent 1

Slow de-correlation phenomenon

(P. L. Ferrari and I. Corwin, P.L. Ferrari, S. Péché)

Simulations

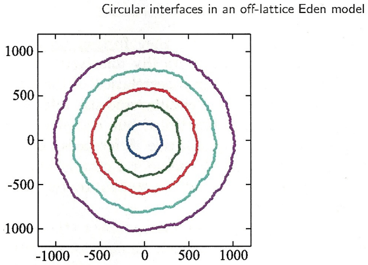

Figure: Clusters at different times in an Eden model
(K. A. Takeuchi, "Statistics of circular interface fluctuations in an off-lattice Eden model", J. Stat. Mech:Th. and Exp., 2012)

Simulations

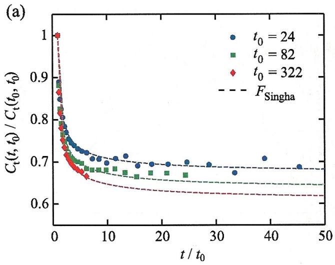

Figure: Temporal correlation function,
(K. A. Takeuchi "Statistics of circular interface fluctuations in an off-lattice Eden model", J. Stat. Mech:Th. and Exp., 2012)

Slow de-correlation

Zero temperature Brownian semi-discrete directed polymer

Last-passage time

$$H(\mu, n) = \sup_{0=\tau_0<\tau_1<\cdots<\tau_n=\mu} \sum_{i=1}^{n} B_i(\tau_i) - B_i(\tau_{i-1}).$$

Distributed as the largest eigenvalue of a GUE-matrix

$$\mathbb{P}[H(\mu, n) \le \xi] = \frac{1}{Z_{\mu,n}} \int_{(-\infty,\xi]^n} \prod_{1\le j<k\le n} (x_k - x_j)^2 \prod_{j=1}^{p} e^{-\frac{x_j^2}{2\mu}} \, d^n x.$$

Zero temperature Brownian semi-discrete directed polymer

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Limit of $G(m, n)$

$$\frac{G([\mu T], n) - \frac{q}{1-q}[\mu T]}{\frac{\sqrt{q}}{1-q}\sqrt{T}} \to H(\mu, n)$$

in distribution as $T \to \infty$.

Main theorem

Theorem

Let $0 < t_1 < t_2$, $\eta_1, \eta_2, \nu_1, \nu_2 \in \mathbb{R}$ *be given. Set*

$$\alpha = (t_1/(t_2 - t_1))^{1/3}.$$

Introduce the scaling, $i = 1, 2$,

$$\mu_i = t_i M - \nu_i (t_i M)^{2/3}, \; n_i = t_i M + \nu_i (t_i M)^{2/3}, \quad \xi_i = 2t_i M + (\eta_i - \nu_i^2)(t_i M)^{1/3}.$$

$i = 1, 2$. *With this scaling,*

$$\lim_{M \to \infty} \mathbb{P}[H(\mu_1, n_1) \leq \xi_1, H(\mu_2, n_2) \leq \xi_2] = F_{tt}(\eta_1, \eta_2; \alpha, \nu_1, \nu_2).$$

There is an explicit, but rather complicated, formula for the *two-time distribution* F_{tt}.

A different formula was derived non-rigorously by V. Dotsenko using the replica method.

A Markov chain

Let
$$\mathbf{G}(m) = (G(m,1), \dots, G(m,n))$$
and
$$w_m(x) = (1-q)^m \binom{x+m-1}{x} q^x 1(x \geq 0).$$
For $\mathbf{x}, \mathbf{y} \in \{\mathbf{x} \in \mathbb{Z}^n \,;\, x_1 \leq x_2 \leq \cdots \leq x_n\}$ and $m > \ell \geq 0$,
$$\mathbb{P}[\mathbf{G}(m) = \mathbf{y} \,|\, \mathbf{G}(\ell) = \mathbf{x}] = \det \left(\Delta^{j-i} w_{m-\ell}(y_j - x_i) \right)_{1 \leq i,j \leq n}.$$
In particular
$$\mathbb{P}[\mathbf{G}(m) = \mathbf{x}] = \det \left(\Delta^{j-i} w_m(x_j) \right)_{1 \leq i,j \leq n}.$$

(Inspired by similar results by J. Warren.)

Joint distribution

We get the formula

$$
\begin{aligned}
&\mathbb{P}[G(m_1, n_1) \le v_1, G(m_2, n_2) \le v_2] \\
&= \sum_{u=-\infty}^{v_1} \sum_{\substack{x \in W_{n_2} \\ x_{n_1} = u}} \sum_{\substack{y \in W_{n_2} \\ y_{n_2} \le v_2}} \det\left(\Delta^{j-i} w_{m_1}(x_j)\right)_{1 \le i,j \le n_2} \det\left(\Delta^{j-i} w_{m_2 - m_1}(y_j - x_i)\right)_{1 \le i,j \le n_2},
\end{aligned}
$$

Joint distribution

We get the formula

$$\mathbb{P}[G(m_1,n_1)\le v_1, G(m_2,n_2)\le v_2]$$
$$= \sum_{u=-\infty}^{v_1} \sum_{\substack{x\in W_{n_2}\\ x_{n_1}=u}} \sum_{\substack{y\in W_{n_2}\\ y_{n_2}\le v_2}} \det\left(\Delta^{j-i} w_{m_1}(x_j)\right)_{1\le i,j\le n_2} \det\left(\Delta^{j-i} w_{m_2-m_1}(y_j-x_i)\right)_{1\le i,j\le n_2},$$

Into this formula we can insert

$$\Delta^k w_m(x) = \frac{(1-q)^m}{2\pi i}\int_{\gamma_r} \frac{(1-z)^k dz}{(1-qz)^m z^{x+k+1}}.$$

Joint distribution

After a non-trivial computation we obtain

$$\mathbb{P}[G(m_1,n_1)\le v_1, G(m_2,n_2)\le v_2]$$
$$= \sum_{u=-\infty}^{v_1} \frac{(1-q)^{m_2 n_2}(-1)^{n_2(n_2-1)/2}}{(2\pi i)^{2n_2} n_1!^2 (\Delta n)!^2} \int_{\gamma_{s_1}^{n_1}} d^{n_1}z \int_{\gamma_{s_2}^{\Delta n}} d^{\Delta n}z \int_{\gamma_{r_1}^{n_1}} d^{n_1}w \int_{\gamma_{r_2}^{\Delta n}} d^{\Delta n}w$$
$$\times \det\left(z_j^{i-1}\right)_{1\le i,j\le n_2} \det\left(w_j^{i-1}\right)_{1\le i,j\le n_2} \det\left(\frac{1}{w_j-z_i}\right)_{1\le i,j\le n_1} \det\left(\frac{1}{z_j-w_i}\right)_{n_1<i,j\le n_2}$$
$$\times \prod_{j=n_1+1}^{n_2}\frac{1-z_j}{1-w_j}\left(1-\prod_{j=1}^{n_1}\frac{z_j}{w_j}\right)\prod_{j=1}^{n_2}\frac{w_j^{u-v_2-\Delta n}}{z_j^{u+n_1}(1-z_j)^{\Delta n}(1-qz_j)^{m_1}(1-w_j)^{n_1}(1-qw_j)^{\Delta m}},$$

Here $\Delta n = n_2 - n_1$, $\Delta m = m_2 - m_1$, $0 < s_1 < r_1 < 1$, $0 < r_2 < s_2 < 1$.

Two identities

An important ingredient in the above derivation are two symmetrization identities:

$$\sum_{\sigma\in S_n} \operatorname{sgn}(\sigma) \prod_{j=1}^{n} \left(\frac{1-w_{\sigma(j)}}{w_{\sigma(j)}}\right)^{j} \frac{1}{(1-w_{\sigma(1)})(1-w_{\sigma(1)}w_{\sigma(2)})\cdots(1-w_{\sigma(1)}\cdots w_{\sigma(n)})}$$

$$= (-1)^{\frac{n(n-1)}{2}} \prod_{j=1}^{n} \frac{1}{w_j^n} \det\left(w_j^{i-1}\right)_{1\le i,j\le n},$$

$\big($Tracy-Widom ASEP identity$\big)$ and

$$\sum_{\sigma_1,\sigma_2\in S_n} \operatorname{sgn}(\sigma_1\sigma_2) \prod_{j=1}^{n} \left(\frac{w_{\sigma_2(j)}(1-z_{\sigma_1(j)})}{z_{\sigma_1(j)}(1-w_{\sigma_2(j)})}\right)^{j}$$

$$\times \frac{1}{\left(1-\frac{z_{\sigma_1(1)}}{w_{\sigma_2(1)}}\right)\left(1-\frac{z_{\sigma_1(1)}z_{\sigma_1(2)}}{w_{\sigma_2(1)}w_{\sigma_2(2)}}\right)\cdots\left(1-\frac{z_{\sigma_1(1)}\cdots z_{\sigma_1(n)}}{w_{\sigma_2(1)}\cdots w_{\sigma_2(n)}}\right)}$$

$$= \prod_{j=1}^{n} \frac{w_j^{n+1}(1-z_j)^n}{z_j^n(1-w_j)^n} \det\left(\frac{1}{w_j-z_i}\right)_{1\le i,j\le n}.$$

Limit to the Brownian directed polymer

Taking the appropriate limit to the Brownian directed polymer we obtain

$$\frac{\partial}{\partial \xi_1}\mathbb{P}\left[H(\mu_1, n_1) \leq \xi_1, H(\mu_2, n_2) \leq \xi_2\right] = \left.\frac{\partial}{\partial h}\right|_{h=0} Q(h),$$

where we recall that

$$H(\mu, n) = \sup_{0=\tau_0<\tau_1<\cdots<\tau_n=\mu} \sum_{i=1}^{n} B_i(\tau_i) - B_i(\tau_{i-1}).$$

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where

$$Q(h) = \frac{(-1)^{\frac{n_2(n_2-1)}{2}}}{(2\pi i)^{2n_2} n_1!\Delta n!}$$
$$\times \int_{\Gamma_{d_1}^{n_1}} d^{n_1}z \int_{\Gamma_{d_2}^{\Delta n}} d^{\Delta n}z \int_{\Gamma_{d_3}^{n_1}} d^{n_1}w \int_{\Gamma_{d_4}^{\Delta n}} d^{\Delta n}w \det\left(z_j^{i-1}\right)_{1\le i,j\le n_2} \det\left(w_j^{i-1}\right)_{1\le i,j\le n_2}$$
$$\times \prod_{j=1}^{n_1} \frac{e^{\frac{1}{2}\mu_1 z_j^2-\xi_1 z_j+\frac{1}{2}\Delta\mu w_j^2-\Delta\xi w_j}}{z_j^{\Delta n} w_j^{n_1}}\left(\frac{1}{z_j-w_j}-h\right)\prod_{j=n_1+1}^{n_2}\frac{e^{\frac{1}{2}\mu_1 z_j^2-\xi_1 z_j+\frac{1}{2}\Delta\mu w_j^2-\Delta\xi w_j}}{z_j^{\Delta n-1} w_j^{n_1+1}(w_j-z_j)}.$$

$d_1 < d_3 < 0$, $d_4 < d_2 < 0$ and $\Delta\xi = \xi_2-\xi_1$, $\Delta\mu = \mu_2-\mu_1$.

Limit to the Brownian directed polymer

$$Q(h) = \frac{(-1)^{\frac{n_2(n_2-1)}{2}}}{(2\pi i)^{2n_2} n_1! \Delta n!}$$
$$\times \int_{\Gamma_{d_1}^{n_1}} d^{n_1} z \int_{\Gamma_{d_2}^{\Delta n}} d^{\Delta n} z \int_{\Gamma_{d_3}^{n_1}} d^{n_1} w \int_{\Gamma_{d_4}^{\Delta n}} d^{\Delta n} w \det\left(z_j^{i-1}\right)_{1\le i,j\le n_2} \det\left(w_j^{i-1}\right)_{1\le i,j\le n_2}$$
$$\times \prod_{j=1}^{n_1} \frac{e^{\frac{1}{2}\mu_1 z_j^2 - \xi_1 z_j + \frac{1}{2}\Delta\mu w_j^2 - \Delta\xi w_j}}{z_j^{\Delta n} w_j^{n_1}} \left(\frac{1}{z_j - w_j} - h\right) \prod_{j=n_1+1}^{n_2} \frac{e^{\frac{1}{2}\mu_1 z_j^2 - \xi_1 z_j + \frac{1}{2}\Delta\mu w_j^2 - \Delta\xi w_j}}{z_j^{\Delta n - 1} w_j^{n_1+1}(w_j - z_j)}.$$

$d_1 < d_3 < 0$, $d_4 < d_2 < 0$ and $\Delta\xi = \xi_2 - \xi_1$, $\Delta\mu = \mu_2 - \mu_1$.

In order to understand the asymptotics of $Q(h)$ we need to rewrite it further so that we can do an appropriate expansion.

Limit to the Brownian directed polymer

$$
\begin{aligned}
Q(h) = & \frac{1}{(2\pi i)^{4n_2} n_1!(\Delta n)!} \int_{\Gamma_{d_1}^{n_1}} d^{n_1} z \int_{\Gamma_{d_2}^{\Delta n}} d^{\Delta n} z \int_{\Gamma_{d_3}^{n_1}} d^{n_1} w \int_{\Gamma_{d_4}^{\Delta n}} d^{\Delta n} w \int_{\gamma_{\tau_1}^{n_2}} d^{n_2} \zeta \int_{\gamma_{\tau_2}^{n_2}} d^{n_2} \omega \\
& \times \det \left(\frac{1}{\zeta_j^i} \right)_{1 \le i,j \le n_2} \det \left(\frac{1}{\omega^{n_2+1-i}} \right)_{1 \le i,j \le n_2} \\
& \times \prod_{j=1}^{n_1} \frac{z_j^{n_1} w_j^{\Delta n} e^{\frac{1}{2}\mu_1 z_j^2 - \xi_1 z_j + \frac{1}{2}\Delta\mu w_j^2 - \Delta\xi w_j}}{e^{\frac{1}{2}\mu_1 \zeta_j^2 - \xi_1 \zeta_j + \frac{1}{2}\Delta\mu \omega_j^2 - \Delta\xi \omega_j} (\zeta_j - z_j)(\omega_j - w_j)} \left(\frac{1}{z_j - w_j} - h \right) \\
& \times \prod_{j=n_1+1}^{n_2} \frac{z_j^{n_1+1} w_j^{\Delta n - 1} e^{\frac{1}{2}\mu_1 z_j^2 - \xi_1 z_j + \frac{1}{2}\Delta\mu w_j^2 - \Delta\xi w_j}}{e^{\frac{1}{2}\mu_1 \zeta_j^2 - \xi_1 \zeta_j + \frac{1}{2}\Delta\mu \omega_j^2 - \Delta\xi \omega_j} (w_j - z_j)(\zeta_j - z_j)(\omega_j - w_j)},
\end{aligned}
$$

where

$$
d_1 < d_3 < -\max(\tau_1, \tau_2) < 0 \ , \ d_4 < d_2 < -\max(\tau_1, \tau_2) < 0.
$$

Expansion – Analogous simpler computation

Largest eigenvalue $\lambda_{\max}$ of a GUE matrix

$$\mathbb{P}_{\mathrm{GUE}\,(n)}[\lambda_{\max} \leq a] = \frac{1}{Z_n} \int_{(-\infty,a]^n} \det\left(x_j^{i-1}\right)^2 \prod_{j=1}^{n} e^{-x_j^2}\, d^n x$$

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$$= \frac{1}{n!} \int_{(-\infty,a]^n} \det\left(h_{i-1}(x_j)\right) \det\left(h_{i-1}(x_j) e^{-x_j^2}\right) \, d^n x,$$

where $h_m(x)$ is the m:th normalized Hermite polynomial.

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where $h_m(x)$ is the m:th normalized Hermite polynomial.

$$= \det\left(\delta_{ij} - \int_a^{\infty} h_{i-1}(x) h_{j-1}(x) e^{-x^2} \, dx\right).$$

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$$= \det\left(\delta_{ij} - \int_a^\infty h_{i-1}(x) h_{j-1}(x) e^{-x^2} \, dx\right).$$

Can be expanded in a Fredholm expansion.

Expansion – Analogous simpler computation

Contour integral formulas

$$h_n(x) = \frac{\sqrt{n!}}{2^{n/2}\pi^{1/4}2\pi i} \int_{|w|=1} \frac{e^{-w^2+2xw}}{w^{n+1}} dw$$

$$e^{-x^2} h_n(x) = \frac{2^{n/2}}{\sqrt{n!}\pi^{3/4} i} \int_{\mathrm{Re}\, z=-2} z^n e^{z^2-2xz} dz.$$

Expansion – Analogous simpler computation

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$$e^{-x^2}h_n(x) = \frac{2^{n/2}}{\sqrt{n!}\pi^{3/4}i}\int_{\mathrm{Re}\,z=-2} z^n e^{z^2-2xz}dz.$$

Insert into

$$\mathbb{P}_{\mathrm{GUE}\,(n)}[\lambda_{\max} \le a] = \frac{1}{n!}\int_{(-\infty,a]^n}\det\left(h_{i-1}(x_j)\right)\det\left(h_{i-1}(x_j)e^{-x_j^2}\right)\, d^n x.$$

Expansion – Analogous simpler computation

We get

$$\mathbb{P}_{\mathrm{GUE}\,(n)}[\lambda_{\max} \leq a] = \frac{2^n}{n!(2\pi i)^{2n}} \int_{(-\infty,a]^n} d^n x \int_{\mathrm{Re}\, z_j = -2} d^n z \int_{|w_j|=1} d^n w$$

$$\det\left(z_j^{i-1}\right) \det\left(\frac{1}{w_j^i}\right) \prod_{j=1}^{n} \frac{e^{z_j^2 - 2x_j z}}{e^{w_j^2 - 2x_j w_j}}.$$

Expansion – Analogous simpler computation

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$$\det\left(z_j^{i-1}\right) \det\left(\frac{1}{w_j^i}\right) \prod_{j=1}^{n} \frac{e^{z_j^2 - 2x_j z}}{e^{w_j^2 - 2x_j w_j}}.$$

Now,

$$\int_{-\infty}^{a} e^{2x_j(w_j - z_j)} dx_j = \frac{e^{2a(w_j - z_j)}}{2(w_j - z_j)}$$

since $\mathrm{Re}\,(w_j - z_j) > 0$.

Expansion – Analogous simpler computation

We obtain

$$\mathbb{P}_{\mathrm{GUE}\,(n)}[\lambda_{\max} \le a]$$

$$= \frac{1}{n!(2\pi i)^{2n}} \int_{\mathrm{Re}\, z_j = -2} d^n z \int_{|w_j|=1} d^n w \det\left(z_j^{i-1}\right) \det\left(\frac{1}{w_j^i}\right) \prod_{j=1}^{n} \frac{e^{z_j^2 - 2az}}{(w_j - z_j) e^{w_j^2 - 2aw_j}}.$$

Expansion – Analogous simpler computation

Expanding the w-determinant and using symmetry yields

$$\mathbb{P}_{\mathrm{GUE}\,(n)}[\lambda_{\max} \leq a]$$
$$= \frac{1}{(2\pi i)^{2n}} \int_{\mathrm{Re}\, z_j = -2} d^n z \int_{|w_j|=1} d^n w \det\left(z_j^{i-1}\right) \prod_{j=1}^{n} \frac{e^{z_j^2 - 2az}}{w_j^j (w_j - z_j) e^{w_j^2 - 2aw_j}}.$$

Expansion – Analogous simpler computation

$$\mathbb{P}_{\mathrm{GUE}\,(n)}[\lambda_{\max} \leq a]$$
$$= \frac{1}{(2\pi i)^{2n}} \int_{\mathrm{Re}\, z_j=-2} d^n z \int_{|w_j|=1} d^n w \det\left(z_j^{i-1}\right) \prod_{j=1}^{n} \frac{e^{z_j^2-2az}}{w_j^j (w_j - z_j) e^{w_j^2-2aw_j}}.$$

We can now expand the z-determinant.

Expansion – Analogous simpler computation

$$\mathbb{P}_{\mathrm{GUE}\,(n)}[\lambda_{\max} \le a]$$
$$= \frac{1}{(2\pi i)^{2n}} \int_{\mathrm{Re}\, z_j = -2} d^n z \int_{|w_j|=1} d^n w \det\left(z_j^{i-1}\right) \prod_{j=1}^{n} \frac{e^{z_j^2 - 2az}}{w_j^j (w_j - z_j) e^{w_j^2 - 2aw_j}}.$$

We can now expand the z-determinant.

$$\mathbb{P}_{\mathrm{GUE}\,(n)}[\lambda_{\max} \le a]$$
$$= \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) \prod_{j=1}^{n} \frac{1}{(2\pi i)^2} \int_{\mathrm{Re}\, z = -2} dz \int_{|w|=1} dw \frac{z^{\sigma(j)-1} e^{z^2 - 2az}}{w^j (w - z) e^{w^2 - 2aw}}.$$

Expansion – Analogous simpler computation

$$\mathbb{P}_{\mathrm{GUE}\,(n)}[\lambda_{\max} \le a]$$
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An asymptotic analysis at the edge requires the z-contour to be on the right side of the w-contour.

Expansion – Analogous simpler computation

$$\mathbb{P}_{\mathrm{GUE}\,(n)}[\lambda_{\max} \le a]$$
$$= \sum_{\sigma\in S_n} \operatorname{sgn}(\sigma) \prod_{j=1}^{n} \frac{1}{(2\pi i)^2} \int_{\mathrm{Re}\, z=-2} dz \int_{|w|=1} dw \, \frac{z^{\sigma(j)-1} e^{z^2-2az}}{w^j (w-z) e^{w^2-2aw}}.$$

An asymptotic analysis at the edge requires the z-contour to be on the right side of the w-contour.

$$\mathbb{P}_{\mathrm{GUE}\,(n)}[\lambda_{\max} \le a]$$
$$= \sum_{\sigma\in S_n} \operatorname{sgn}(\sigma) \prod_{j=1}^{n} \Bigg(\frac{1}{2\pi i} \int_{|w|=1} w^{\sigma(j)-j-1}\, dw$$
$$- \frac{1}{(2\pi i)^2} \int_{\mathrm{Re}\, z=2} dz \int_{|w|=1} dw \, \frac{z^{\sigma(j)-1} e^{z^2-2az}}{w^j (w-z) e^{w^2-2aw}} \Bigg).$$

Expansion – Analogous simpler computation

This gives

$$\mathbb{P}_{\mathrm{GUE}\,(n)}[\lambda_{\max} \leq a]$$
$$= \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) \prod_{j=1}^{n} \left(\delta_{j,\sigma(j)} - \frac{1}{(2\pi i)^2} \int_{\operatorname{Re} z = 2} dz \int_{|w|=1} dw \, \frac{z^{\sigma(j)-1} e^{z^2 - 2az}}{w^j (w - z) e^{w^2 - 2aw}} \right).$$

Expand the product.

Expansion – Analogous simpler computation

This gives

$$\mathbb{P}_{\mathrm{GUE}\,(n)}[\lambda_{\max} \le a]$$
$$= \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) \prod_{j=1}^{n} \left(\delta_{j,\sigma(j)} - \frac{1}{(2\pi i)^2} \int_{\mathrm{Re}\, z=2} dz \int_{|w|=1} dw \, \frac{z^{\sigma(j)-1} e^{z^2-2az}}{w^j (w-z) e^{w^2-2aw}} \right).$$

Expand the product.

This is nothing but

$$\mathbb{P}_{\mathrm{GUE}\,(n)}[\lambda_{\max} \le a] = \det \left(\delta_{ij} - \int_a^\infty h_{i-1}(x) h_{j-1}(x) e^{-x^2} \, dx \right).$$

in a new form.

The two-time distribution

Recall

$$\lim_{M\to\infty} \mathbb{P}[H(\mu_1, n_1) \le \xi_1, H(\mu_2, n_2) \le \xi_2] = F_{\mathrm{tt}}(\eta_1, \eta_2; \alpha).$$

where

$$\mu_i = t_i M = n_i = t_i M, \quad \xi_i = 2t_i M + \eta_i (t_i M)^{1/3},$$

$i = 1, 2$, and

$$\alpha = (t_1/(t_2 - t_1))^{1/3}.$$

The two-time distribution

We have the formula

$$
\begin{aligned}
&F_{\mathrm{tt}}(\eta_1^*, \eta_2; \alpha) \\
&= F_2(\eta_2) - \sum_{r,s,t=0}^{\infty} \frac{1}{(r!)^2 s! t!} \\
&\times \int_{\eta_1^*}^{\infty} d\eta_1 \int_{(-\infty,0]^r} d^r x \int_{(-\infty,0]^s} d^s x' \int_{[0,\infty)^r} d^r y \int_{[0,\infty)^t} d^t y' \, W^{(1)}_{r,s,r,t}(\mathbf{x}, \mathbf{x}', \mathbf{y}, \mathbf{y}') \\
&- \sum_{r=1}^{\infty} \sum_{s,t=0}^{\infty} \frac{1}{r!(r-1)!s!t!} \\
&\times \int_{\eta_1^*}^{\infty} d\eta_1 \int_{(-\infty,0]^r} d^r x \int_{(-\infty,0]^s} d^s x' \int_{[0,\infty)^{r-1}} d^{r-1} y \int_{[0,\infty)^t} d^t y' \, W^{(2)}_{r,s,r-1,t}(\mathbf{x}, \mathbf{x}', \mathbf{y}, \mathbf{y}'),
\end{aligned}
$$

The two-time distribution

Block determinant

$$W^{(1)}_{r_1,s,r_2,t}(\mathbf{x},\mathbf{x}',\mathbf{y},\mathbf{y}') = \begin{vmatrix} \psi(\mathbf{x},\mathbf{x}) & \psi(\mathbf{x},\mathbf{x}') & \psi(\mathbf{x},0) & \psi(\mathbf{x},\mathbf{y}) & \psi(\mathbf{x},\mathbf{y}') \\ \phi(\mathbf{x}',\mathbf{x}) & \phi(\mathbf{x}',\mathbf{x}') & \phi(\mathbf{x}',0) & \phi(\mathbf{x}',\mathbf{y}) & \phi(\mathbf{x}',\mathbf{y}') \\ \psi(0,\mathbf{x}) & \psi(0,\mathbf{x}') & \psi(0,0) & \psi(0,\mathbf{y}) & \psi(0,\mathbf{y}') \\ \phi(\mathbf{y},\mathbf{x}) & \phi(\mathbf{y},\mathbf{x}') & \phi(\mathbf{y},0) & \phi(\mathbf{y},\mathbf{y}) & \phi(\mathbf{y},\mathbf{y}') \\ \psi(\mathbf{y}',\mathbf{x}) & \psi(\mathbf{y}',\mathbf{x}') & \psi(\mathbf{y}',0) & \psi(\mathbf{y}',\mathbf{y}) & \psi(\mathbf{y}',\mathbf{y}') \end{vmatrix}$$

$\mathbf{x} \in \mathbb{R}^{r_1}$, $\mathbf{x}' \in \mathbb{R}^{s}$, $\mathbf{y} \in \mathbb{R}^{r_2}$, $\mathbf{y}' \in \mathbb{R}^{t}$ and $0 \in \mathbb{R}$.

$W^{(2)}_{r_1,s,r_2,t}$ is very similar (ψ in middle row replaced by ϕ).

The two-time distribution

The functions ϕ and ψ are defined by

$$\phi(x,y) = \phi_1(x,y) + 1(y \geq 0)\phi_2(x,y) - 1(x < 0)\phi_3(x,y),$$

and

$$\psi(x,y) = -\psi_1(x,y) - 1(y > 0)\phi_2(x,y) + 1(x \leq 0)\phi_3(x,y),$$

where

$$\phi_1(x,y) = -\alpha \int_0^\infty K_{\mathrm{Ai}}\left(\eta_1 - \tau, \eta_1 - y\right) K_{\mathrm{Ai}}\left(\Delta\eta + \alpha\tau, \Delta\eta + \alpha x\right) d\tau,$$

$$\psi_1(x,y) = \alpha \int_0^\infty K_{\mathrm{Ai}}\left(\eta_1 + \tau, \eta_1 - y\right) K_{\mathrm{Ai}}\left(\Delta\eta - \alpha\tau, \Delta\eta + \alpha x\right) d\tau,$$

$$\phi_2(x,y) = \alpha K_{\mathrm{Ai}}\left(\Delta\eta + \alpha x, \Delta\eta + \alpha y\right),$$

$$\phi_3(x,y) = K_{\mathrm{Ai}}\left(\eta_1 - x, \eta_1 - y\right).$$

and

$$\Delta\eta = \eta_2 \left(\frac{t_2}{\Delta t}\right)^{1/3} - \eta_1 \left(\frac{t_1}{\Delta t}\right)^{1/3}.$$

Thank you!