On the sup-norm problem for arithmetic hyperbolic 3-manifolds

Gergely Harcos

Alfréd Rényi Institute of Mathematics
http://www.renyi.hu/~gharcos/

29 September 2014
Analytic Number Theory Workshop
University of Oxford

Arithmetic Quantum Chaos (1 of 3)

Consider a freely moving particle on a compact manifold M.

- In classical mechanics, the particle corresponds to an orbit of the geodesic flow on the unit cotangent bundle S^*M.
- In quantum mechanics, the particle corresponds to a solution $\psi : M \times \mathbb{R} \to \mathbb{C}$ of the Schrödinger equation $\triangle\psi + i\frac{\partial\psi}{\partial t} = 0$. $|\psi(m,t)|^2$ is the probability density of the particle. Any nice solution is a linear combination of the stationary waves $(m,t) \mapsto \phi(m)e^{-i\lambda t}$, where $\phi : M \to \mathbb{C}$ satisfies $\triangle\phi + \lambda\phi = 0$.

Assume that M has negative sectional curvature.

- The geodesic flow on S^*M is ergodic (Anosov–Sinai 1967).
- $|\phi(m)|^2 d\mathrm{vol}(m) \to d\mathrm{vol}(m)$ holds for almost all ϕ (Schnirelman 1974, Colin de Verdière 1985, Zelditch 1987).
- The limit should hold for all ϕ (QUE, Rudnick–Sarnak 1994).
- Any weak star limit of the measures $|\phi(m)|^2 d\mathrm{vol}(m)$ has a lift to S^*M which has positive entropy (Anantharaman 2008).

Arithmetic Quantum Chaos (2 of 3)

Arithmetic Quantum Unique Ergodicity Conjecture

Let M be a compact Riemannian manifold of negative sectional curvature. Assume that $M = \Gamma \backslash S$, where S is a globally symmetric space and $\Gamma \leqslant \mathrm{Isom}^+(S)$ is an arithmetic subgroup of isometries. Let $\phi : M \to \mathbb{C}$ run through a complete orthonormal sequence of Hecke eigenforms. Then the probability measures $|\phi(m)|^2 d\mathrm{vol}(m)$ tend in the weak star topology to the uniform measure $d\mathrm{vol}(m)$.

- AQUE is true for compact arithmetic hyperbolic surfaces $\Gamma \backslash \mathcal{H}^2$ (Lindenstrauss 2006), and also for the modular surface $\mathrm{SL}_2(\mathbb{Z}) \backslash \mathcal{H}^2$ (Soundararajan 2010). In these cases, GRH implies an optimal rate of convergence (Watson 2001).
- The conjecture generalizes to higher rank. AQUE is true for $S = \mathrm{PGL}_n(\mathbb{R}) / \mathrm{PO}_n(\mathbb{R})$, n prime (Silberman–Venkatesh 2007).
- Hecke operators are key in all these results.

Arithmetic Quantum Chaos (3 of 3)

Theorem (Sarnak 2004)

Let $M = \Gamma\backslash S$, where S is a Riemannian globally symmetric space and $\Gamma \leqslant \mathrm{Isom}^+(S)$ is a co-compact discrete subgroup of isometries. Let $\phi : M \to \mathbb{C}$ be an L^2-normalized joint eigenfunction of the invariant differential operators on S. If λ denotes the Laplacian eigenvalue of ϕ, then

$$\|\phi\|_\infty \ll_{S,\Gamma} \lambda^{(\dim S - \mathrm{rank}\, S)/4}.$$

Problem

Fix S. Assume that Γ is arithmetic and ϕ is a Hecke eigenform.

1. *Estimate $\|\phi\|_\infty$ in terms of λ.*
2. *Estimate $\|\phi\|_\infty$ in terms of Γ.*
3. *Examine what happens when $M = \Gamma\backslash S$ is not compact.*

Results for the sup-norm problem on arithmetic manifolds

group	eigenvalue aspect	level aspect
$GL_2(\mathbb{R})$	Iwaniec–Sarnak 95	Abbes–Ullmo 95, Michel–Ullmo 98
	Rudnick 05, Xia 07	Jorgenson–Kramer 04
	Friedman–Jorgenson–Kramer 14^+	Blomer–Holowinsky 10
		Templier 10, Helfgott–Ricotta 11
	Templier 14^+	Harcos–Templier 13
	Das–Sengupta 14^+, Steiner 14^+	Ye 14^+, Kiral 14^+, Saha 14^+
	Sarnak 04, Milićević 10	Lau 10, Templier 14, Saha 14^+
$GL_2(\mathbb{C})$	Koyama 95	
	Blomer–Harcos–Milićević 14^+	Blomer–Harcos–Milićević 14^+
	Rudnick–Sarnak 94, Milićević 11	
$SO_n(\mathbb{R})$	VanderKam 97, Blomer–Michel 13	Blomer–Michel 13
$Sp_4(\mathbb{R})$	Blomer–Pohl 14^+	
$GL_n(\mathbb{R})$	Holowinsky–Ricotta–Royer 14^+	
	Blomer–Maga 14^+, Marshall 14^+	
	Brumley–Templier 14^+	

Hyperbolic plane and hyperbolic space

- Consider $\mathcal{H}^2 := \{x + yi :\ x \in \mathbb{R},\ y > 0\}$ with the $\mathrm{GL}_2(\mathbb{R})$-action

$$gz = (az+b)(cz+d)^{-1} \qquad g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2^+(\mathbb{R})$$

$$gz = -\bar{z} \qquad g = \begin{pmatrix} -1 & \\ & 1 \end{pmatrix}$$

- Consider $\mathcal{H}^3 := \{z + rj :\ z \in \mathbb{C},\ r > 0\}$ with the $\mathrm{GL}_2(\mathbb{C})$-action

$$gP = (aP+b)(cP+d)^{-1} \qquad g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2^+(\mathbb{C})$$

$$gP = P \qquad g = \begin{pmatrix} a & \\ & a \end{pmatrix} \in Z(\mathbb{C})$$

- $\boxed{\mathcal{H}^2 \cong \mathrm{Z}(\mathbb{R})\backslash \mathrm{GL}_2(\mathbb{R}) / \mathrm{O}_2(\mathbb{R})}$ and $\boxed{\mathcal{H}^3 \cong \mathrm{Z}(\mathbb{C})\backslash \mathrm{GL}_2(\mathbb{C}) / \mathrm{U}_2(\mathbb{C})}$

Results for $\mathcal{H}^2$ and square-free level $N \in \mathbb{Z}$

Congruence subgroup

$$\Gamma_0(N) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) \;\middle|\; c \equiv 0 \pmod{N} \right\}$$

Theorem (Iwaniec–Sarnak 1995)

$$\|\phi\|_\infty \ll_{N,\varepsilon} \lambda^{\frac{5}{24}+\varepsilon}$$

Theorem (Blomer–Holowinsky 2010)

$$\|\phi\|_\infty \ll_{\lambda,\varepsilon} N^{-\frac{25}{914}+\varepsilon}$$

Theorem (Templier 2013)

$$\|\phi\|_\infty \ll_{\varepsilon} \lambda^{\frac{5}{24}+\varepsilon} N^{-\frac{1}{6}+\varepsilon}$$

Results for $\mathcal{H}^3$ and square-free level $N \in \mathbb{Z}[i]$

Congruence subgroup

$$\Gamma_0(N) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}[i]) \;\middle|\; c \equiv 0 \pmod{N} \right\}$$

Theorem (Koyama 1994)

$$\|\phi\|_\infty \ll_{N,\varepsilon} \lambda^{\frac{37}{76}+\varepsilon}$$

Theorem (Blomer–Harcos–Milićević 2013)

$$\|\phi\|_\infty \ll_\varepsilon (\lambda|N|)^\varepsilon \min(\lambda^{\frac{5}{12}}, \lambda^{\frac{1}{2}}|N|^{-\frac{1}{3}})$$

Theorem (Blomer–Harcos–Milićević 2013)

$$\|\phi\|_\infty \ll_\varepsilon \lambda^{\frac{4}{9}+\varepsilon}|N|^{-\frac{1}{9}+\varepsilon}$$

The five pillars of the proof

1. Amplification method (Duke–Friedlander–Iwaniec)
2. Pretrace formula (Selberg)
3. Arithmetic symmetries (Hecke, Atkin–Lehner)
4. Geometry of numbers (Gauss, Minkowski)
5. Diophantine approximation (Dirichlet)

The Key Lemma

Theorem (Blomer–Harcos–Milićević 2013)

Let ϕ be an L^2-normalized Hecke–Maass newform on $\mathcal{H}^3$ of square-free level $N \in \mathbb{Z}[i]$. Then

$$|\phi(P)| \ll_\varepsilon (\lambda|N|)^\varepsilon \min(\lambda^{\frac{5}{12}}, \lambda^{\frac{1}{2}}|N|^{-\frac{1}{3}}), \qquad P \in \mathcal{H}^3.$$

Key Lemma

The supremum of $|\phi(P)|$ is attained at a point $P = z + rj \in \mathcal{H}^3$ such that the associated lattice $\mathbb{Z}[i] + \mathbb{Z}[i]P \subset \mathbb{R} + \mathbb{R}i + \mathbb{R}j + \mathbb{R}k$ and its successive minima $m_1 \leqslant m_2 \leqslant m_3 \leqslant m_4$ satisfy:

1. $|N|^{-\frac{1}{2}} \leqslant m_1 = m_2 \leqslant m_3 = m_4$
2. $m_1 m_2 m_3 m_4 \asymp r^2$ *and* $r \gg |N|^{-1}$
3. *In any ball of radius R the number of lattice points is*

$$\ll 1 + R^2|N| + R^4 r^{-2}.$$

Reduction to a matrix counting problem

Notation

$$X \preccurlyeq Y \quad \stackrel{\text{def}}{\Longleftrightarrow} \quad X \ll_\varepsilon Y(\lambda|N|)^\varepsilon$$

Proposition

There is $\lambda^{-1} \leqslant \delta \preccurlyeq 1$ *such that*

$$|\phi(P)|^2 \preccurlyeq \frac{\sqrt{\lambda}}{\sqrt{\delta}} \left(r^2\delta + \frac{1}{L} + \frac{M(P,L,\delta)}{L^3} + \frac{\widetilde{M}(P,L,\delta)}{L^4} \right),$$

where

- $M(P,L,\delta)$ *is the number of matrices* $\gamma \in \mathrm{M}_2(\mathbb{Z}[i])$ *such that*

  $$\cosh(\mathrm{dist}(\gamma P, P)) \leqslant 1 + \delta,$$

  the lower left entry of γ *is nonzero and divisible by* N, *and* $\det\gamma = l_1 l_2$ *with split primes* $l_1, l_2 \asymp \sqrt{L}$ *from the first octant;*
- $\widetilde{M}(P,L,\delta)$ *is same with* $\det\gamma = l_1^2 l_2^2$ *(instead of* $\det\gamma = l_1 l_2$*).*

The distance condition

We write $P = z + rj \in \mathcal{H}^3$ and $\det\gamma = l \asymp \sqrt{\mathcal{L}}$.

$$\cosh(\operatorname{dist}(\gamma P, P)) \leqslant 1 + \delta, \qquad \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{M}_2(\mathbb{Z}[i])$$

$$\Updownarrow$$

$$\|a'P + b' - Pc'P - Pd'\|^2 \leqslant 2r^2\delta, \quad \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} := \frac{1}{\sqrt{l}} \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

Geometric principle

Asssume $\cosh(\operatorname{dist}(\gamma P, P)) \leqslant 1 + \delta$. *Then*

$$\|cP + d\|, \ \|cP - a\| = |l|^{1/2} + O(\mathcal{L}^{1/4}\sqrt{\delta}).$$

Moreover, if c is fixed and the angle of l varies only $O(\sqrt{\delta})$, then

1. *$(a - d)z + b$ lies in a disk of radius $O(r\mathcal{L}^{1/4}\sqrt{\delta})$;*
2. *$a + d$ lies in a rectangle $\mathcal{R}$ of size $O(\mathcal{L}^{1/4}\sqrt{\delta}) \times \widetilde{O}(\mathcal{L}^{1/4})$;*
3. *$a - d$ lies in the rotated rectangle $2cz + i\mathcal{R}$.*

$M(P, L, \delta)$ for all δ, and $\widetilde{M}(P, L, \delta)$ for $\delta \gg L^{-4}$

Goal

$$M(P, L, \delta) \preccurlyeq L^2 + L^4 \min(\sqrt{\delta}, |N|^{-2})$$

$$\widetilde{M}(P, L, \delta) \preccurlyeq L^3 + L^6 \min(\sqrt{\delta}, |N|^{-2})$$

Idea ($c \rightsquigarrow a - d \rightsquigarrow b \rightsquigarrow a + d$)

For fixed c, the lattice point $(a-d)P + b \in \mathbb{Z}[i] + \mathbb{Z}[i]P$ lies in a small ball. If, in addition, l lies in a small angular sector, then $a-d$ and $a+d$ lie in thin rectangles. Finally, for l square, $a+d$ is essentially determined by $(c, a-d, b)$ via

$$(a-d)^2 + 4bc = (a + d - 2\sqrt{l})(a + d + 2\sqrt{l}).$$

Idea ($c \rightsquigarrow a \rightsquigarrow l \rightsquigarrow d$)

If c and a are fixed, then the rational integer $|l|^2$ lies in a small interval. If l is also fixed, then d is restricted to a small disk.

$\widetilde{M}(P, L, \delta)$ for $\delta \ll L^{-4}$ (1 of 2)

Idea

In this case $l = \det \gamma$ *is a square, so* $\lambda := \sqrt{l}$ *is a Gaussian integer. The lattice triangle* $0, a+d, \lambda$ *has tiny height by the distance condition, so its area is zero. Arithmetic in Gaussian integers shows that* $a + d \in \{0, \pm\lambda, \pm 2\lambda\}$, *hence* λ *essentially determines* $a + d$.

Idea

Approximate Nz *by a Gaussian fraction:*

$$Nz = \frac{p}{q} + O\left(\frac{1}{|q|L^2}\right),$$

where $p, q \in \mathbb{Z}[i]$ *and* $1 \leqslant |q| \leqslant L^2$ *and* $(p, q) = 1$. *Proceed differently for* $|q| \leqslant L$ *and for* $|q| > L$.

$\widetilde{M}(P, L, \delta)$ for $\delta \ll L^{-4}$ (2 of 2)

Idea (for $|q| \leqslant L$)

Write $c = Nc'$. The matrix $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{M}_2(\mathbb{Z}[i])$ is essentially determined by the product $(2c'p - aq + dq)\overline{\lambda}$ which lies in a rectangle of size $O(1 + |q|L^2\sqrt{\delta}) \times O(|q|L^2)$. Hence

$$\widetilde{M}(P, L, \delta) \ll (1 + |q|L^2\sqrt{\delta})|q|L^2 \ll L^3 + L^6\sqrt{\delta}.$$

Idea (for $|q| > L$)

The matrix $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{M}_2(\mathbb{Z}[i])$ is essentially determined by the product $c'(\overline{a} - \overline{a_{c'}})$ which lies in $\ll (1 + |q|L^2\sqrt{\delta})|q|/|(q, \overline{q})|$ translates of a 2-dimensional lattice with minimal length $\gg |q|/|(q, \overline{q})|$ and covolume $\gg |q|^2/|(q, \overline{q})|$. Hence

$$\widetilde{M}(P, L, \delta) \ll (1 + |q|L^2\sqrt{\delta})\frac{|q|}{|(q, \overline{q})|}\left(1 + \frac{L^2|(q, \overline{q})|}{|q|} + \frac{L^4|(q, \overline{q})|}{|q|^2}\right)$$

$$\ll L^3 + L^6\sqrt{\delta}.$$