Multilevel Dyson Brownian motion and its edge limits

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Gaussian Unitary Ensemble

Gaussian Unitary Ensemble of rank N and variance t is the distribution on the set of $N \times N$ *Hermitian* matrices with density

$$\rho(M) \sim \exp\left(-\frac{\mathrm{Trace}(M^2)}{2t}\right).$$

Alternatively, $M = (X + X^*)/2$, where X is $N \times N$ matrix of i.i.d. Gaussian random variables (real and imaginary parts of variance t).

Eigenvalues are real.

GUE–eigenvalues density (H. Weyl, E. Cartan, 20s)

$$\rho(x_1, \dots, x_N) \sim \prod_{i<j} (x_i - x_j)^2 \prod_{i=1}^{N} \exp\left(-\frac{(x_i)^2}{2t}\right).$$

Gaussian β Ensemble

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Turning to other natural matrix models we can twist the density changing the parameter $\beta = 2$ to a different one.

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$$\rho(x_1, \ldots, x_N) \sim \prod_{i<j} (x_j - x_i)^{\textcircled{1}} \prod_{i=1}^{N} \exp\left(-\frac{(x_i)^2}{2t}\right)$$

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- $\beta = 1$ corresponds to real symmetric matrices

Gaussian β Ensemble

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Turning to other natural matrix models we can twist the density changing the parameter $\beta = 2$ to a different one.

- $\beta = 1$ corresponds to real symmetric matrices
- $\beta = 2$ corresponds to complex Hermitian matrices

Gaussian β Ensemble

$$\rho(x_1, \dots, x_N) \sim \prod_{i<j} (x_j - x_i)^{4} \prod_{i=1}^{N} \exp\left(-\frac{(x_i)^2}{2t}\right)$$

Turning to other natural matrix models we can twist the density changing the parameter $\beta = 2$ to a different one.

- $\beta = 1$ corresponds to real symmetric matrices
- $\beta = 2$ corresponds to complex Hermitian matrices
- $\beta = 4$ corresponds to quaternion Hermitian matrices.

Gaussian β Ensemble

$$\rho(x_1, \dots, x_N) \sim \prod_{i<j} (x_j - x_i)^{\beta} \prod_{i=1}^{N} \exp\left(-\frac{(x_i)^2}{2t}\right)$$

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General β parameter suggests a statistical mechanics interpretation as the inverse temperature.

General β ensembles can be also obtained as eigenvalues of tridiagonal matrices and as limits of Macdonald processes.

F_β as an RM edge limit

$$\rho(x_1, \ldots, x_N) \sim \prod_{i<j}(x_j - x_i)^\beta \prod_{i=1}^{N} \exp\left(-\frac{(x_i)^2}{2t}\right).$$

$\mathcal{X}_1^N(t; \beta)$ denotes the smallest eigenvalue.

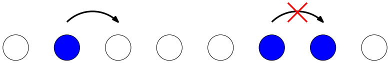
Theorem. (Tracy–Widom, Forrester, Ramirez–Rider–Virag)

$$-\frac{2N + \mathcal{X}_1^N\left(\frac{\beta N}{2}; \beta\right)}{N^{1/3}} \xrightarrow{N\to\infty} F_\beta,$$

F_β is β**–Tracy–Widom distribution**.

F_2 as TASEP large time limit

Consider Totally Asymmetric Simple Exclusion Process



Particle independently jump to the right by one after exponential waiting time, unless they are blocked.

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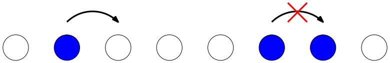

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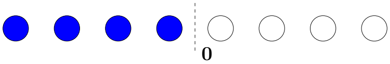
$H(x,t)$ it the number of particles to the right of x after time t.



Theorem. (Johansson) For step initial condition

$$\frac{t/4 - H(0,t)}{t^{1/3}} \xrightarrow{t\to\infty} F_2.$$

F_β is $\beta = 2$ **Tracy–Widom distribution**.

Between RM and IPS

GUE

smallest eigenvalue

$$-\frac{2N + \mathcal{X}_1^N(N;2)}{N^{1/3}} \xrightarrow{N\to\infty} F_2,$$

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Our topic for today:

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1. Why are the behaviors in these seemingly unrelated systems so similar?

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Our topic for today:

1. Why are the behaviors in these seemingly unrelated systems so similar?
2. Is there any extension of the interplay between interacting particle systems and random matrices to general values of β?

Dyson Brownian Motion

To connect to TASEP we must have a dynamics on eigenvalues.

Dyson Brownian Motion

The projection of the independent Brownian Motion evolution on matrix elements onto the spectrum gives for $\beta = 1, 2, 4$

$$\mathrm{d}X_i^N(t) = \frac{\beta}{2}\sum_{j\neq i}\frac{1}{X_i^N(t) - X_j^N(t)}\mathrm{d}t + \mathrm{d}B_i(t), \quad 1 \leq i \leq N,$$

which makes sense for any $\beta > 0$ and agrees with $G\beta E$

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$$\rho(x_1, \dots, x_N) \sim \prod_{i<j} (x_j - x_i)^\beta \prod_{i=1}^{N} \exp\left(-\frac{(x_i)^2}{2t}\right).$$

This is **β Dyson Brownian Motion**. We see no locality in the interactions yet — seems to be far from TASEP.

Alternative dynamics on *GUE*

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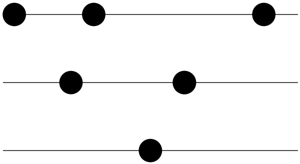

Interlacing condition reads $W_i^{j+1}(t) \leq W_i^j(t) \leq W_{i+1}^{j+1}(t)$ almost surely for each t.

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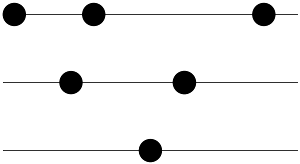

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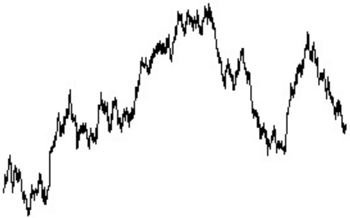
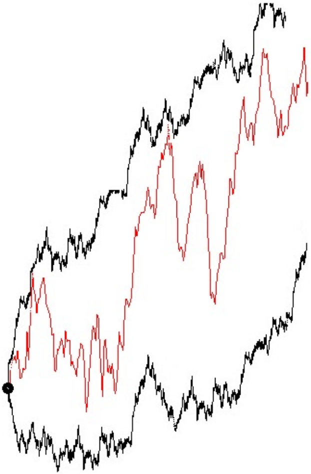
Inductive definition.
Let $W_1^1(t)$ be the standard **Brownian motion.**



(GUE of rank 1 is a standard Gaussian.)

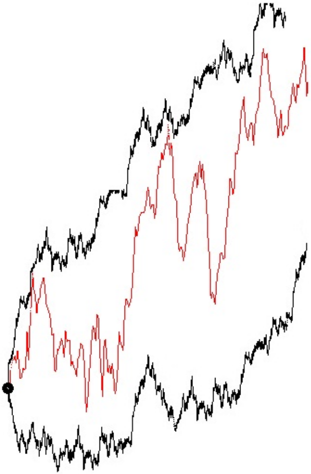
Alternative dynamics on *GUE*



Given the trajectory of $W_1^1(t)$ (in red) we define $W_1^2(t)$ and $W_2^2(t)$ as two (independent) Brownian motions **reflected** by $W_1^1(t)$ (and, thus, separated by this trajectory).

Example. W_t reflected by horizontal line $x = 0$ is $|W_t|$.

Alternative dynamics on GUE



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More generally, $W_i^j(t)$ is a Brownian motion **reflected** by trajectories $W_{i-1}^{j-1}(t)$ and $W_i^{j-1}(t)$ and lying in between.

Alternative dynamics on GUE

The most interesting case is *zero* initial condition, i.e. $W_i^j(0) = 0$.

Theorem 1. (Warren, 2005) For any fixed N the restriction of the dynamics on $W_i^N(t)$, $i = 1, \ldots, N$ is ($\beta = 2$) **Dyson Brownian Motion**.

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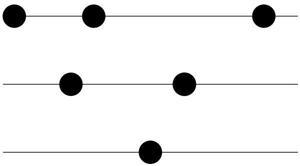
Theorem 2. (Warren, 2005) For any fixed t the distribution of $N(N+1)/2$ particles $W_i^j(t)$, $i = 1, \dots, j$, $j = 1, \dots, N$ is the distribution of **GUE–corners** process of variance t.



Joint distribution of eigenvalues of top–left $k \times k$ corners from GUE random matrix.

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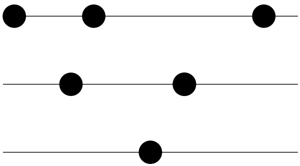

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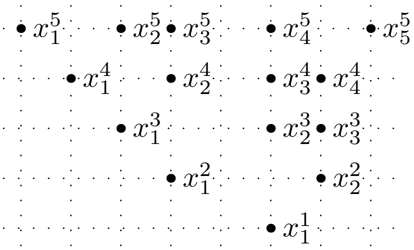
The interactions in the Warren process are local. Limit of TASEP?

Multilevel TASEP

Discrete counterpart of the Warren process (Borodin–Ferrari)



Continuous time dynamics $Y(t)$ on particle configurations with integer coordinates subject to interlacing $x_{i-1}^j < x_{i-1}^{j-1} \leq x_i^j$.

Each particle has an exponential clock of rate 1. All clocks are independent. When the clock rings particle attempts to jump to the right.

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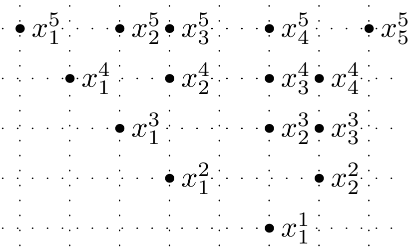

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The interlacing conditions are preserved by the rule "if higher, then lighter".

Multilevel TASEP

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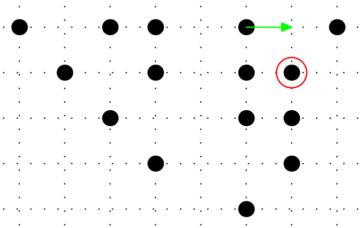

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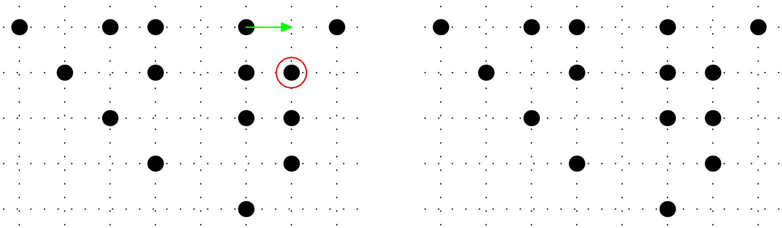

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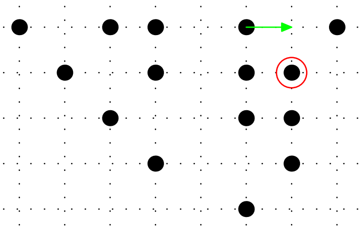
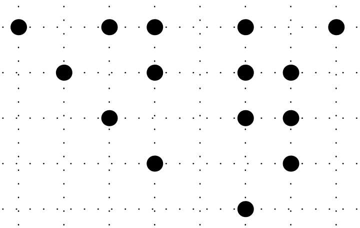
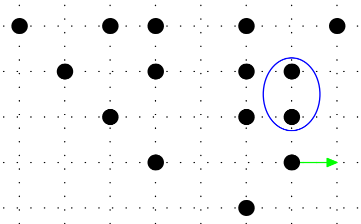




Push:



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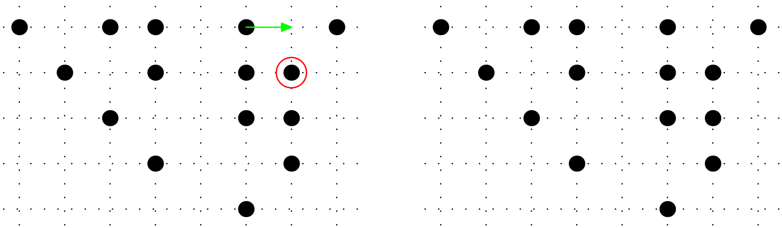

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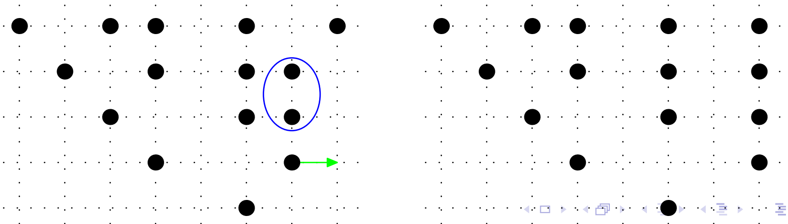

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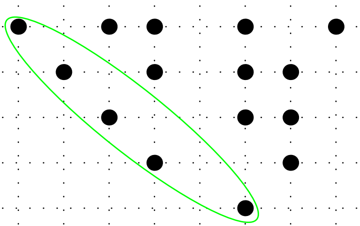
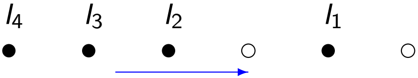
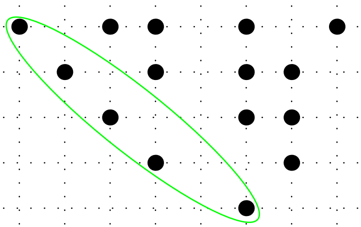
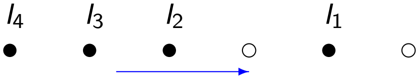




Totally Asymmetric Simple Exclusion Process.

Multilevel TASEP





Totally Asymmetric Simple Exclusion Process.

Evolution of rightmost particles is Long Range TASEP (or PushASEP).

Multilevel TASEP $\to$ Warren process

We could even start with *any* $N(N+1)/2$ random walks with unit jumps and construct a similar dynamics through block/push interaction between particles.

Theorem. (Gorin–Shkolnikov, 2012) Warren process $W(t)$ serves as a **general scaling limit** for various dynamics on interlacing particles with block/push interactions. In particular, for multilevel TASEP $Y(t)$ as $S \to \infty$ we have

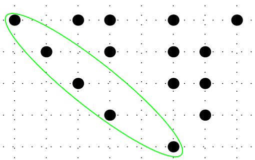
$$\frac{Y(tS) - tS}{\sqrt{tS}} \to W(t).$$

(convergence of the laws of $N(N+1)/2$-dimensional trajectories).

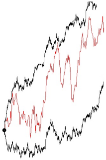
In particular, fixed time distribution of $Y(t)$ and similar processes converges to GUE–corners process.

Conjecture. Restriction on unit jumps can be removed.

Conclusion on TASEP vs GUE



TASEP is a part of a larger dynamics — **multilevel TASEP**



Along the lines of classical CLT's its diffusive scaling limit is the **Warren process** of reflected interlacing Brownian Motions.

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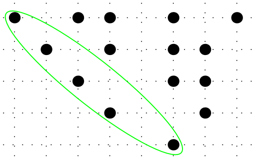

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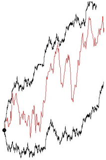

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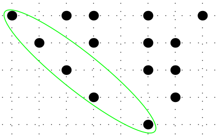
That's why **Tracy–Widom F_2** governs the asymptotic of both.

General β ensembles vs IPS?



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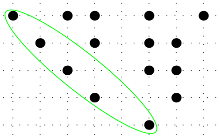

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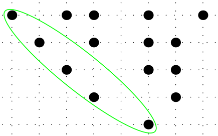

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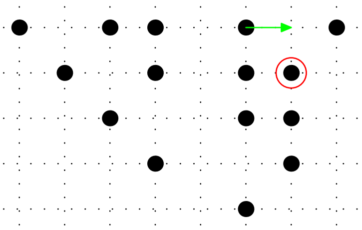
But let us try to mimic the just presented $\beta = 2$ path.

Jack polynomials deformation



Multilevel TASEP can be constructed through a probabilistic procedure based on the coupling of Markov chains (Diaconis–Fill, Borodin–Ferrari). The key *input* is given by **Schur symmetric polynomials**

$$s_\lambda(x_1, \ldots, x_N) = \frac{\det_{i,j=1}^{N}[x_i^{\lambda_j+N-j}]}{\prod_{i<j}(x_i - x_j)}.$$

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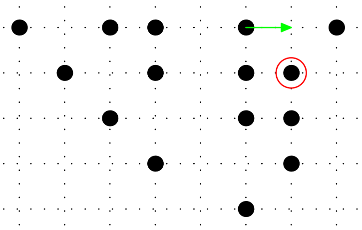

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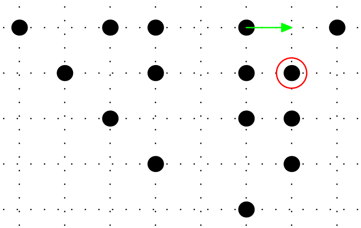
The latter admit a 1–parametric deformation **Jack polynomials** $J_\lambda^\theta(x_1,\dots,x_N)$.

$$J_\lambda^1(x_1,\dots,x_N) = s_\lambda(x_1,\dots,x_N)$$

Jack polynomials deformation



Multilevel TASEP can be constructed through a probabilistic procedure based on the coupling of Markov chains (Diaconis–Fill, Borodin–Ferrari). The key *input* is given by **Schur symmetric polynomials**

$$s_\lambda(x_1,\dots,x_N) = \frac{\det_{i,j=1}^N [x_i^{\lambda_j+N-j}]}{\prod_{i<j}(x_i - x_j)}.$$

The latter admit a 1–parametric deformation **Jack polynomials** $J_\lambda^\theta(x_1,\dots,x_N)$.

$$J_\lambda^1(x_1,\dots,x_N) = s_\lambda(x_1,\dots,x_N)$$

Diffusive scaling limit gives dynamics on $\beta = 2\theta$ Gaussian ensemble.

Dynamics on $G\beta E$–corners processes

Here is the **diffusive limit.**

Theorem. (Gorin–Shkolnikov–2014) For any N and $\beta > 2$ the following system of SDEs has a unique (interlacing) solution

$$\mathrm{d}X_i^k(t) = \left(\sum_{m \neq i} \frac{1-\beta/2}{X_i^k(t) - X_m^k(t)} - \sum_{m=1}^{k-1} \frac{1-\beta/2}{X_i^k(t) - X_m^{k-1}(t)} \right) \mathrm{d}t + \mathrm{d}W_i^k, \quad 1 \leq i \leq k \leq N$$

with W_i^k being independent standard Brownian motions

$$\begin{array}{ccccccc}
X_1^4 & & X_2^4 & & X_3^4 & & X_4^4 \\
& X_1^3 & & X_2^3 & & X_3^3 & \\
& & X_1^2 & & X_2^2 & & \\
& & & X_1^1 & & &
\end{array}$$

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with W_i^k being independent standard Brownian motions

Theorem. (Gorin–Shkolnikov–2014) Suppose $\beta \geq 4$ and $X_i^j(0) = 0$. Then for each k the evolution of the row $X_1^k(t), \dots, X_k^k(t)$ is **β–Dyson Brownian Motion.**

$$\mathrm{d}X_i^N(t) = \frac{\beta}{2} \sum_{j \neq i} \frac{1}{X_i^N(t) - X_j^N(t)} \mathrm{d}t + \mathrm{d}B_i(t), \quad 1 \leq i \leq N,$$

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Theorem. (Gorin–Shkolnikov–2014) Suppose $\beta \ge 4$ and $X_i^j(0) = 0$. For each fixed t the the array $X_i^k(t)$, $1 \le i \le k \le N$ is distributed as **$G\beta E$–corners** process of variance t.

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{pmatrix}$$

General β extension of the joint distribution for the eigenvalues of corners of Gaussian real/complex/quaternion self–adjoint matrices at $\beta = 1, 2, 4$.

Dynamics on $G\beta E$–corners processes

Theorem. (Gorin–Shkolnikov–2014) For any N and $\beta > 2$ the following system of SDEs has a unique (interlacing) solution

$$\mathrm{d}X_i^k(t) = \left(\sum_{m \neq i} \frac{1-\beta/2}{X_i^k(t) - X_m^k(t)} - \sum_{m=1}^{k-1} \frac{1-\beta/2}{X_i^k(t) - X_m^{k-1}(t)} \right) \mathrm{d}t + \mathrm{d}W_i^k, \quad 1 \le i \le k \le N.$$

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$$\prod_{i<j} (x_j^N - x_i^N) \prod_{i=1}^{N} \exp\left(-\frac{1}{2t} \left(x_i^N\right)^2 \right) \times \prod_{k=1}^{N-1} \prod_{1 \le i < j \le k} (x_j^k - x_i^k)^{2-\beta} \prod_{a=1}^{k} \prod_{b=1}^{k+1} |x_a^k - x_b^{k+1}|^{\beta/2-1}.$$

Dynamics on $G\beta E$–corners processes

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Remark 1. At $\beta < 4$ particles start to collide (cf. Bessel process).
Remark 2. At $\beta \leq 2$ SDE makes no sense. At $\beta = 2$ **local time terms** are necessary. For $\beta < 2$ the correct SDE is unclear.
Remark 3. Construction through **discrete approximations** related to Jack polynomials works for any $\beta > 0$. For small β we do not have any good identification for the desired continuous object.

Dynamics on $G\beta E$–corners processes

Theorem. (Gorin–Shkolnikov–2014) For any N and $\beta > 2$ the following system of SDEs has a unique (interlacing) solution

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This is the **multilevel Dyson Brownian motion** — general β analogue of the Warren process.

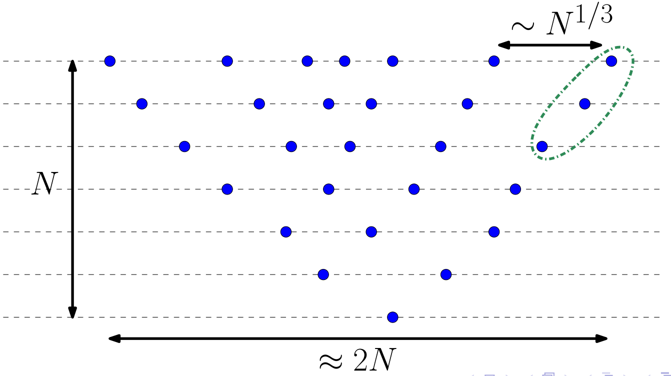
- Connection to random matrices is still there.
- Locality of the interactions seems to disappear at $\beta \neq 2$.

How to revive the locality?

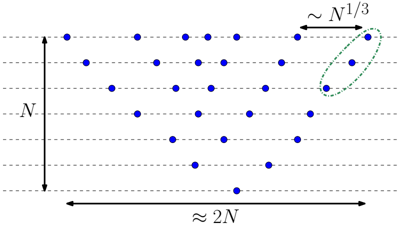
Edge limit of multilevel Dyson Brownian motion

$$\mathrm{d}X_i^k(t) = \left(\sum_{m \neq i} \frac{1-\beta/2}{X_i^k(t) - X_m^k(t)} - \sum_{m=1}^{k-1} \frac{1-\beta/2}{X_i^k(t) - X_m^{k-1}(t)} \right) \mathrm{d}t + \mathrm{d}W_i^k$$

Our way to make locality reappear is through an edge limit.
We wait time $T = \frac{2N}{\beta}$ and observe:



Edge limit of multilevel Dyson Brownian motion



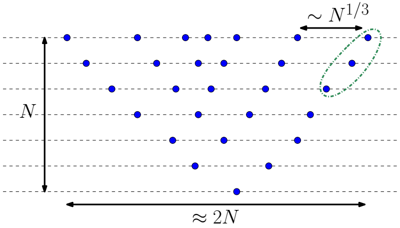
Theorem. (Gorin–Shkolnikov–2014) For any $\beta \geq 4$ and $k = 1, 2, \ldots$ the distribution of the process

$$\Big(X_N^N\Big(\frac{2N}{\beta}+t\Big) - X_{N-1}^{N-1}\Big(\frac{2N}{\beta}+t\Big), \ldots, X_{N-k+1}^{N-k+1}\Big(\frac{2N}{\beta}+t\Big) - X_{N-k}^{N-k}\Big(\frac{2N}{\beta}+t\Big)\Big), \quad t \geq 0$$

converges weakly to that of a stationary Markov process

$$\big(R_1(t), R_2(t), \ldots, R_k(t)\big), \quad t \geq 0.$$

Description of the edge limit



Theorem. (Gorin–Shkolnikov–2014) For any $\beta \geq 1$ and $k = 1, 2, \ldots,$ $t \in \mathbb{R}$ the *fixed time* spacings

$$\left(X_N^N\left(\frac{2N}{\beta} + t\right) - X_{N-1}^{N-1}\left(\frac{2N}{\beta} + t\right), \ldots, \right.$$

$$\left. X_{N-k+1}^{N-k+1}\left(\frac{2N}{\beta} + t\right) - X_{N-k}^{N-k}\left(\frac{2N}{\beta} + t\right) \right)$$

converge weakly to **independent Gamma distributions** of density

$$\frac{1}{\Gamma(\beta/2)} \left(\frac{\beta}{2}\right)^{\beta/2} x^{\beta/2-1}\, e^{-\frac{\beta}{2}x}, \quad x > 0.$$

Description of the edge limit

Theorem. (Gorin–Shkolnikov–2014) For any $\beta \geq 4$ and any k

$$\Big(X_N^N\Big(\frac{2N}{\beta}+t\Big) - X_{N-1}^{N-1}\Big(\frac{2N}{\beta}+t\Big), \dots, X_{N-k+1}^{N-k+1}\Big(\frac{2N}{\beta}+t\Big) - X_{N-k}^{N-k}\Big(\frac{2N}{\beta}+t\Big)\Big)$$

$$\longrightarrow \big(Z_1^{(k)}(t) - Z_2^{(k)}(t), \dots, Z_k^{(k)}(t) - Z_{k+1}^{(k)}(t)\big), \quad t \geq 0,$$

where $Z_i^{(k)}$ solve the SDEs with **local interactions**

$$\mathrm{d}Z_i^{(k)}(t) = \frac{(\beta/2-1)\,\mathrm{d}t}{Z_i^{(k)}(t) - Z_{i+1}^{(k)}(t)} + \mathrm{d}B_i(t), \quad i = 1, 2, \dots, k,$$

$$\mathrm{d}Z_{k+1}^{(k)}(t) = \mathrm{d}t + \mathrm{d}B_{k+1}(t),$$

$B_1, B_2, \dots, B_{k+1}$ are i.i.d. Brownian motions.

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$B_1, B_2, \dots, B_{k+1}$ are i.i.d. Brownian motions.

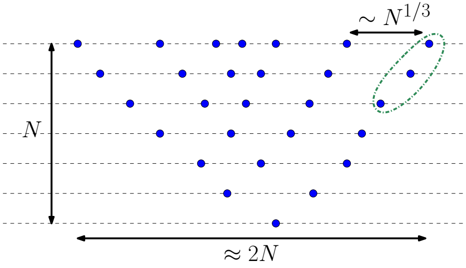
Remark. Different k spacings are **compatible** with each other.

Description of the edge limit

Theorem. (Gorin–Shkolnikov–2014) Limiting spacings $Z_i^{(k)} - Z_{i+1}^{(k)}$ solve the SDEs with **local interactions**

$$\mathrm{d}Z_i^{(k)}(t) = \frac{(\beta/2-1)\,\mathrm{d}t}{Z_i^{(k)}(t) - Z_{i+1}^{(k)}(t)} + \mathrm{d}B_i(t), \quad i = 1, 2, \ldots, k,$$

$$\mathrm{d}Z_{k+1}^{(k)}(t) = \mathrm{d}t + \mathrm{d}B_{k+1}(t),$$



Therefore, an interacting particle system with local interactions appears at the **edge** of general β Dyson Brownian Motion.

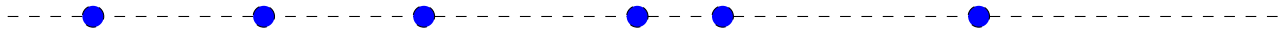
Description of the edge limit

Limit is a stationary dynamics with i.i.d. Gamma spacings.

$$\text{distribution}\big(Z_i(t) - Z_{i+1}(t)\big) : \quad \frac{1}{\Gamma(\beta/2)} \left(\frac{\beta}{2}\right)^{\beta/2} x^{\beta/2-1}\, e^{-\frac{\beta}{2}x}, \quad x > 0.$$

$$\mathrm{d}Z_i(t) = \frac{(\beta/2 - 1)\,\mathrm{d}t}{Z_i(t) - Z_{i+1}(t)} + \mathrm{d}B_i(t).$$

And evolution of k adjacent spacings is Markovian of explicit form.



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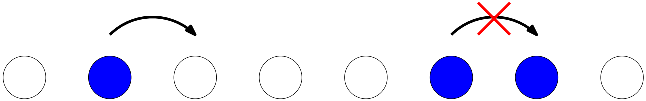
And evolution of k adjacent spacings is Markovian of explicit form.

This is very similar to the stationary version of TASEP on $\mathbb{Z}$ with Bernoully invariant measure — where spacings are i.i.d. geometric.

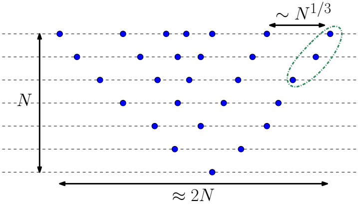


(But note that TASEP corresponds to $\beta = 2$, while we have $\beta \geq 4$)

New type of edge limit in random matrices

By-product for classical Gaussian ensembles of random matrices:

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{pmatrix}$$



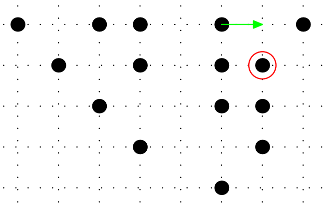
Corollary. Consider a G[O/U/S]E random matrix of size $N \times N$, such that the variance of its diagonal elements is $2N/\beta$. Write $\lambda^{(k)}(N)$ for the largest eigenvalue of the $(N+1-k) \times (N+1-k)$ top-left submatrix. Then,

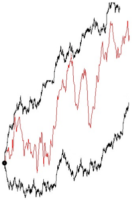
$$\left(\lambda^{(1)}(N) - \lambda^{(2)}(N),\ \lambda^{(2)}(N) - \lambda^{(3)}(N),\ \dots,\ \lambda^{(K)}(N) - \lambda^{(K+1)}(N)\right)$$

converge to i.i.d. Gamma distributions with density

$$\frac{1}{\Gamma(\beta/2)} \left(\frac{\beta}{2}\right)^{\beta/2} x^{\beta/2-1}\, e^{-\frac{\beta}{2}x}, \qquad x > 0.$$

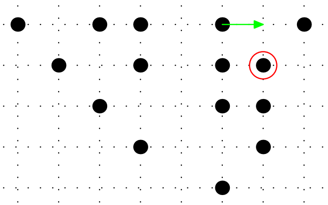
A little bit about proofs

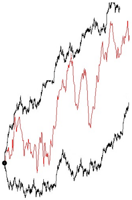




Proof of convergence of multilevel TASEP to reflected interlacing Brownian Motions of Warren.

A little bit about proofs

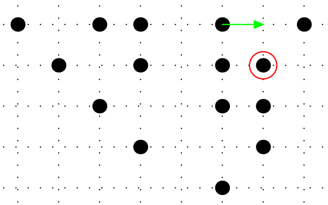


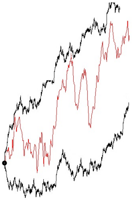


Proof of convergence of multilevel TASEP to reflected interlacing Brownian Motions of Warren.

- Identification of multilevel TASEP with image of independent multidimensional noise under *deterministic* (Skorohod–type) reflection map.

A little bit about proofs





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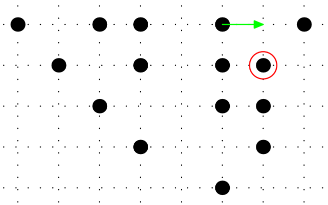

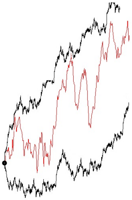

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- (Burdzy–Kang–Ramanan–2009) prove continuity of such maps.

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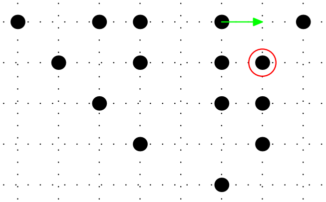

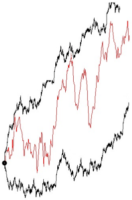

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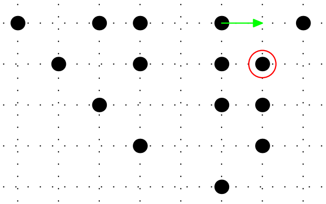
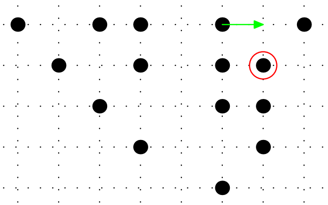
- (Burdzy–Kang–Ramanan–2009) prove continuity of such maps.
- Convergence of Poissonian process to Brownian Motion yields the result.

A little bit about proofs



Diffusive scaling limit for the deformation related to Jack polynomials

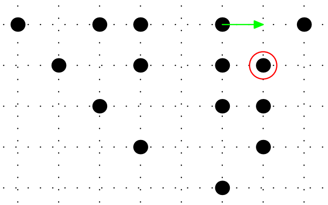
A little bit about proofs



Diffusive scaling limit for the deformation related to Jack polynomials

- Construction of prelimit-objects via Markov operators build out of Jack–polynomials and Diaconis–Fill coupling idea.

A little bit about proofs



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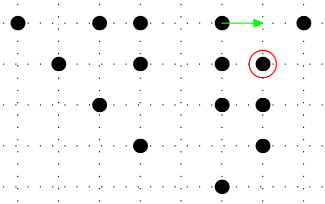

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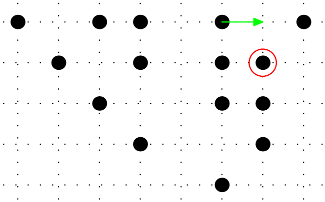
- Identification of the limiting points is done via martingale problem techniques in the spirit of Stroock and Varadhan.

A little bit about proofs



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- Tightness via explicit estimates on transitional probabilities.
- Identification of the limiting points is done via martingale problem techniques in the spirit of Stroock and Varadhan.
- Need to use the fact that the limiting diffusion does not hit singularities at the boundaries. This is proved via appropriate Lyapunov functions and a version of the Feller explosion test.

A little bit about proofs

$$\mathrm{d}X_i^k(t) = \left(\sum_{m \neq i} \frac{1-\beta/2}{X_i^k(t) - X_m^k(t)} - \sum_{m=1}^{k-1} \frac{1-\beta/2}{X_i^k(t) - X_m^{k-1}(t)} \right) \mathrm{d}t + \mathrm{d}W_i^k, \quad 1 \le i \le k \le N.$$

Proof of the existence/uniqueness for the limiting SDE.

A little bit about proofs

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- Another Lyapunov function argument shows that many particles do not collide simultaneously.

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Proof of the existence/uniqueness for the limiting SDE.

- Another Lyapunov function argument shows that many particles do not collide simultaneously.
- This allows to use the Girsanov change of measure to reduce (locally in time) the dynamics to independent Bessel process.

A little bit about proofs

$$\mathrm{d}X_i^k(t) = \left(\sum_{m \neq i} \frac{1-\beta/2}{X_i^k(t) - X_m^k(t)} - \sum_{m=1}^{k-1} \frac{1-\beta/2}{X_i^k(t) - X_m^{k-1}(t)} \right) \mathrm{d}t + \mathrm{d}W_i^k, \quad 1 \leq i \leq k \leq N.$$

Proof of the existence/uniqueness for the limiting SDE.

- Another Lyapunov function argument shows that many particles do not collide simultaneously.
- This allows to use the Girsanov change of measure to reduce (locally in time) the dynamics to independent Bessel process.
- Existence/Uniqueness for the latter is well-known.

A little bit about proofs

Proof of convergence of spacing in $G\beta E-$ corners to i.i.d. Gammas.

$$\prod_{i<j}(x_j^N - x_i^N)\prod_{i=1}^{N}\exp\left(-\frac{1}{2t}\left(x_i^N\right)^2\right)$$

$$\times\prod_{k=1}^{N-1}\prod_{1\leq i<j\leq k}(x_j^k - x_i^k)^{2-\beta}\prod_{a=1}^{k}\prod_{b=1}^{k+1}|x_a^k - x_b^{k+1}|^{\beta/2-1}.$$

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- Reduction to two–level (N particles plus $(N-1)$ particles) distributions via Dixon–Anderson integration formula.
- Analysis of the resulting density relies on convergence

$$\sqrt{\frac{\beta t}{2N}}\sum_{i=1}^{N-1}\frac{1}{X_N^N - X_i^N}\to 1,\quad \sqrt{\frac{\beta t}{2N}}\frac{1}{X_N^N - X_{N-1}^N}\to 0$$

in a strong sense. We also use large deviations results for Gaussian β–ensembles in the bulk (Bourgade–Erdos–Yau) and for largest eigenvalues (Ledoux–Rider).

A little bit about proofs

Proof of dynamical limit theorem leading to local interactions:

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Two sums tend to the same limit as $N \to \infty$ and cancel out.

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- Actual proof shows tightness through comparisons with Bessel processes and identifies the limit via martingale problem techniques in the spirit of Stroock and Varadhan.

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- At $\beta = 2$ the interplay between random matrices and TASEP can be explained by **multilevel TASEP** and its diffusive scaling limit to **Warren process**, related to GUE.

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- Interactions become **local** after considering specific edge limits.

$$\mathrm{d}Z_i(t) = \frac{(\beta/2 - 1)\,\mathrm{d}t}{Z_i(t) - Z_{i+1}(t)} + \mathrm{d}B_i(t).$$

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- **Open question.** Can we obtain the β–Tracy–Widom distributions F_β in the asymptotic of (this or another) interacting particle system?