

The ternary  
Goldbach problem

Harald Andrés  
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Introduction

Fourier analysis

The circle method

The major arcs

The minor arcs

Conclusion

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September 2014

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## What was known?

From the letters of Leonhard Euler and Christian Goldbach:

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**Ternary, or weak, Golbach conjecture (1742)**  
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## A little more history

Descartes (1630s? 1640s?) - Every integer  $n \geq 1$  is the sum of one, two or three prime numbers (conjecture in a posthumously published manuscript).

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This could have been used to check the ternary conjecture up to, say,  $10^7$ : it is enough to construct a "ladder" of prime numbers  $p_1, p_2, p_3, \dots$  up to  $10^7$  with  $p_{i+1} - p_i \leq 10000$ .

# The twentieth century, and now

**Hardy-Littlewood (1922):** There is a  $C$  such that every odd number  $\geq C$  is the sum of three primes, if we assume the generalized Riemann hypothesis (GRH).

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News?

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News?

Theorem (Helfgott, May 2013)

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every even  $n \geq 2$  is the sum of  $\leq 6$  primes (**Ramaré, 1995**)

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Ternary Goldbach holds for all  $n$  conditionally on the generalized Riemann hypothesis (GRH)  
(**Deshouillers-Effinger-te Riele-Zinoviev, 1997**)



## Ternary Goldbach: bounds for $n \geq C$

It is clear why computer bounds are for  $n \leq c$ , where  $c$  is a constant.

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Why are analytic bounds for  $n \geq C$ , where  $C$  is a constant?

An analytic proof tells you that the number (or: weighted number) of ways to write  $n$  (here, an odd number) in a specified form (here, as the sum of 3 primes) is

main term + error term,

where "main term" is some precise expression  $f(n)$ , and "error term" is something whose absolute value is at most  $g(n)$ .

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## Ternary Goldbach: improvements in $C$

Every odd  $n \geq C$  is the sum of three primes (Vinogradov)

Bounds for  $C$ ?  $C = 3^{3^{15}}$  (Borozdkin, 1956),

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brought it down to  $10^{27}$ .

## Fourier series

A function  $f : \mathbb{R} \rightarrow \mathbb{C}$  with  $f(\alpha + 1) = f(\alpha)$  can be decomposed into a sum of sines and cosines:

$$f(\alpha) = \sum_n a_n e(\alpha n) = \sum_n a_n (\cos(2\pi\alpha n) + i \sin(2\pi\alpha n)),$$

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Example: The sawtooth function

$$f(\alpha) = \begin{cases} \alpha - [\alpha] & \text{if } \{\alpha\} \in [0, 1/2], \\ [\alpha + 1] - \alpha & \text{if } \{\alpha\} \in [1/2, 1]. \end{cases}$$

can be written as

$$f(\alpha) = \sum_{n \text{ odd}} \frac{e(\alpha n)}{\pi^2 n^2} = \sum_{n \text{ odd}} \frac{1}{\pi^2 n^2} \cdot \cos(2\pi\alpha n).$$

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How to determine  $a_n$ ? Fourier inversion theorem:

$$a_n = \int_0^1 f(\alpha) e(\alpha n) d\alpha.$$

## Fourier analysis: the other way around

What happens if we have a function  $f : \mathbb{Z} \rightarrow \mathbb{C}$ ? We can then write it as an *integral*:

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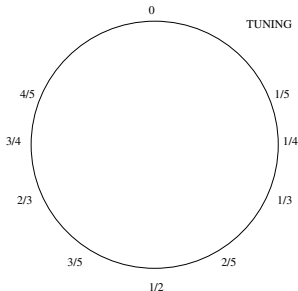
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## The circle method, continued

Hardy and Littlewood used  $f_i(n) = f(n)$ , where  $f(n) = 0$  for  $n$  composite (or  $n \leq 0$ ) and  $f(n) = (\log n)e^{-n/N}$  (where  $N$  will be set later) for  $n$  prime.

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Our task is to show  $(f * f * f)(n) \neq 0$  for  $n > C$  ( $C \sim 10^{30}$ ), since this implies that there are  $m_1, m_2, m_3$  with  $m_1 + m_2 + m_3 = n$  and  $f(m_1), f(m_2), f(m_3) \neq 0$  (and thus  $m_1, m_2, m_3$  prime).

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$$(f * f * f)(n) = \int_0^1 \widehat{f * f * f}(\alpha) e(\alpha n) d\alpha = \int_0^1 (\widehat{f}(\alpha))^3 e(\alpha n) d\alpha.$$

Task: show this last integral is  $> 0$ .

# The basic strategy in the circle method

It will turn out that  $\widehat{f}(\alpha)$  is large when  $\alpha$  is close to a rational  $a/q$  with  $q$  small.



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Idea: estimate  $\widehat{f}(\alpha)$  for  $\alpha$  in the union  $\mathfrak{M}$  of intervals around rationals with small denominators (*major arcs*);

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(This is what can't be done for binary Goldbach: the integral over  $\mathfrak{m}$  is then bigger.)

# The major arcs

To estimate  $\int_{\mathfrak{M}} (\widehat{f}(\alpha))^3 e(-N\alpha)$ , we need to estimate  $\widehat{f}(\alpha)$  for  $\alpha$  near  $a/q$ ,  $q$  small ( $q \leq m(x)$ ).

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We do this by studying  $L(s, \chi)$  for Dirichlet characters mod  $q$ , i.e., characters  $\chi : (\mathbb{Z}/q\mathbb{Z})^* \rightarrow \mathbb{C}$ .

$$L(s, \chi) := \sum_n \chi(n) n^{-s}$$

for  $\Re(s) > 1$ ; this has an analytic continuation to all of  $\mathbb{C}$  (with a pole at  $s = 1$  if  $\chi$  is trivial).

We express  $\widehat{f}(\alpha)$ ,  $\alpha = a/q + \delta/x$ , as a sum of

$$S_{\eta, \chi}(\delta/x, x) = \sum_{n=1}^{\infty} \Lambda(n) \chi(n) e(\delta n/x) \eta(n/x)$$

for  $\chi$  varying among all Dirichlet characters modulo  $q$ .

# The explicit formula

“Explicit formula”:

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- (a) the term  $F_{\delta}(1)x$  appears only for  $\chi$  principal ( $\sim$  trivial),
- (b)  $\rho$  runs over the complex numbers  $\rho$  with  $L(\rho, \chi) = 0$  and  $0 < \Re(\rho) \leq 1$  (called “non-trivial zeroes”),
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Analytic on a strip  $x_0 < \Re(s) < x_1$  in  $\mathbb{C}$ .

It is a Laplace transform (or Fourier transform!) after a change of variables.

# Where are the zeroes of $L(s, \chi)$ ?

Let  $\rho = \sigma + it$  be any non-trivial zero of  $L(s, \chi)$ .

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### What we can also know:

for a given  $\chi$ , we can verify GRH for  $L(s, \chi)$  “up to a height  $T_0$ ”. This means: verify that every zero  $\rho$  with  $|\Im(\rho)| \leq T_0$  satisfies  $\sigma = 1/2$ .

## Verifying GRH up to a given height

For the purpose of proving strong bounds that solve ternary Goldbach, **zero-free regions** are far too weak.

We must rely on **verifying** GRH for several  $L(s, \chi)$ ,  
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For  $\chi$  trivial ( $\chi(x) = 1$ ),  $L(s, \chi) = \zeta(s)$ .

The Riemann hypothesis has been verified up to  
 $|t| \leq 2.4 \cdot 10^{11}$  (**Wedeniwski** 2003),  $|t| \leq 1.1 \cdot 10^{11}$  (**Platt**  
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This has been extended up to  $q \leq 2 \cdot 10^5$ ,  $q$  odd, and  $q \leq 4 \cdot 10^5$ ,  $q$  even ( $|t| \leq 200 + 7.5 \cdot 10^7/q$ ) (**Platt** 2013).

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We recall we must estimate  $\sum_{\rho} F_{\delta}(\rho)x^{\rho}$ , where  $F_{\delta}$  is the Mellin transform of  $\eta(t)e(\delta t)$ .

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The ternary  
Goldbach problem

Harald Andrés  
 Helfgott

# The Gaussian smoothing

Motivation for  $\eta(t) = e^{-t}$  (Hardy-Littlewood)?

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The main term in  $F_\delta(\sigma + i\tau)$  behaves as

$$e^{-\frac{\pi}{4}|\tau|}$$

for  $\delta$  small,  $\tau \rightarrow \pm\infty$ , and as

$$e^{-\frac{1}{2}\left(\frac{|\tau|}{2\pi\delta}\right)^2}$$

for  $\delta$  large,  $\tau \rightarrow \pm\infty$ .

## Major arcs: conclusions

Thus we obtain estimates for  $S_{\eta, \chi}(\delta/x, x)$ , where

$$\eta(t) = g(t)e^{-t^2/2},$$

and  $g$  is any “band-limited” function:

$$g(t) = \int_{-R}^R h(r)t^{-ir} dr$$

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All the rest of the circle must be minor arcs;  $m(x)$  must be a constant  $M$ . (Writer for *Science*: “Muenster cheese” rather than “Swiss cheese”.)

Thus, minor-arc bounds will have to be very strong.

# The new bound for minor arcs

## Theorem (Helfgott, May 2012 – March 2013)

Let  $x \geq x_0$ ,  $x_0 = 2.16 \cdot 10^{20}$ . Let  $2\alpha = a/q + \delta/x$ ,  $\gcd(a, q) = 1$ ,  $|\delta/x| \leq 1/qQ$ , where  $Q = (3/4)x^{2/3}$ . Let  $\eta_2(n) = 4(1_{[1/2,1]} * 1_{[1/2,1]})$ .

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$$\frac{R_{x, \delta_0 q}(\log \delta_0 q + 0.002) + 0.5}{\sqrt{\delta_0 \phi(q)}} + \frac{2.491}{\sqrt{\delta_0 q}}$$

$$+ \frac{2}{\delta_0 \phi(q)} \left( \log \delta_0^{7/4} q^{13/4} + \frac{80}{9} \right)$$

$$+ \frac{2}{\delta_0 q} \left( \log q^{\frac{80}{9}} \delta_0^{\frac{16}{9}} + \frac{111}{5} \right) + 3.2x^{-1/6},$$

where  $\delta_0 = \max(2, |\delta|/4)$ ,

$$R_{x, t_1, t_2} = 0.4141 + 0.2713 \log \left( 1 + \frac{\log 4t_1}{2 \log \frac{9x^{1/3}}{2.004t_2}} \right).$$

## Worst-case comparison

Let us compare the results here (2012-2013) with those of Tao (Feb 2012) for  $q$  highly composite,  $|\delta| < 8$ :

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$10^5$	0.04521	0.34475
$1.5 \cdot 10^5$	0.03820	0.28836
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**Table:** Upper bounds on  $x^{-1}|S_\eta(a/2q, x)|$  for  $q \geq q_0$ ,  $2 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 13|q$ ,  $|\delta| \leq 8$ ,  $x = 10^{25}$ . The trivial bound is 1.

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# The new bounds for minor arcs: ideas

## Qualitative improvements:

- cancellation within Vaughan's identity
- $\delta/x = \alpha - a/q$  is a friend, not an enemy:
  - In type I: (a) decrease of  $\widehat{\eta}$ ,  
change in approximations;  
In type II: scattered input to the large sieve
- relation between the circle method and the large sieve – in its version for primes;
- the benefits of a continuous  $\eta$  (also in Tao, Ramaré),

# Cancellation within Vaughan's identity

Vaughan's identity:

$$\Lambda = \mu_{\leq U} * \log - \Lambda_{\leq V} * \mu_{\leq U} * 1 + 1 * \mu_{> U} * \Lambda_{> V} + \Lambda_{\leq V},$$

where  $f_{\leq V}(n) = f(n)$  if  $n \leq V$ ,  $f_{\leq V}(n) = 0$  if  $n > V$ . (Four summands: **type I**, type I, **type II**, negligible.)

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Alternative would have been: use a log-free formula (e.g. Daboussi-Rivat); proceeding as above seems better in practice.



# How to recover factors of log

Introduction

Fourier analysis

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The major arcs

The minor arcs

Conclusion

In type I sums:

We use cancellation in  $\sum_{n \leq M: d|n} \mu(n)/n$ .

**This is allowed:** we are using only  $\zeta$ , not  $L$ .

**This is explicit:** **Granville-Ramaré, El Marraki, Ramaré.**

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Application of the large sieve for primes.

# The “error” $\delta/x = \alpha - a/q$ is a friend

In type II:

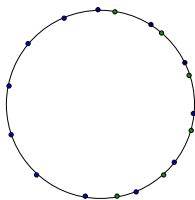
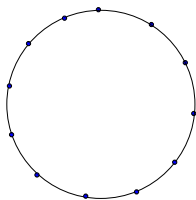
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In type II: the angles  $m\alpha$  are separated by  $\geq \delta/x$  (**even when  $m \geq q$** ). We can apply the large sieve to *all*  $m\alpha$  in one go. We can even use prime support: double scattering, by  $\delta$  and by **Montgomery's** lemma.



# Final result

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All goes well for  $n \geq 10^{27}$  (or well beneath that). As we have seen, the case  $n \leq 10^{27}$  (and in fact  $n \leq 8.8 \cdot 10^{30}$ ) is already done (computation).

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## Theorem (Helfgott, May 2013)

*Every odd number  $n \geq 7$  is the sum of three prime numbers.*