Some new estimates on the Liouville heat kernel

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Outline

Notations

We equip the two dimensional torus $\mathbb{T}$ with:

- $d_{\mathbb{T}}$ standard volume distance and dx volume form
- Δ the Laplace-Beltrami operator on $\mathbb{T}$
- $p_t(x, y)$ the standard heat kernel of the Brownian motion **B** on $\mathbb{T}$

Recall that:

$$p_t(x, y) = \frac{1}{|\mathbb{T}|} + \sum_{n \geq 1} e^{-\lambda_n t} e_n(x) e_n(y)$$

where $(\lambda_n)_{n \geq 1}$ (increasing) eigenvalues and $(e_n)_{n \geq 1}$ (normalized) eigenvectors:

$$-\Delta e_n = 2\pi \lambda_n e_n, \qquad \int_{\mathbb{T}} e_n(x) dx = 0.$$

Log-correlated field X

Notations:

- G standard Green function of the Laplacian Δ:

$$G(x,y) = \sum_{n \geq 1} \frac{1}{\lambda_n} e_n(x) e_n(y)$$

- X GFF on $\mathbb{T}$ under $\mathbb{P}^X$ (expectation $\mathbb{E}^X$):

$$\mathbb{E}^X[X(x)X(y)] = G(x,y) = \ln_+ \frac{1}{d_{\mathbb{T}}(x,y)} + g(x,y)$$

Gaussian multiplicative chaos (Liouville measure)

Gaussian multiplicative chaos associated to X:

$$M_\gamma(dx) = e^{\gamma X(x) - \frac{\gamma^2}{2}\mathbb{E}[X(x)^2]}\, dx.$$

Theorem (Kahane, 1985)

M_γ can be defined by regularizing the field X and a limit procedure. $M_\gamma \neq 0$ if and only if $\gamma < 2$. If $\gamma < 2$, the measure M_γ "lives" almost surely on a set of Hausdorff dimension $2 - \frac{\gamma^2}{2}$ (the set of thick points).

Liouville Brownian motion

Framework:

- Standard Brownian motion $\mathbf{B} = (\mathbf{B}_t)_{t\geq 0}$ on $\mathbb{T}$
- $P^x_{\mathbf{B}}$ (and $E^x_{\mathbf{B}}$) probability (expectation) of $\mathbf{B}$ starting from x.
- $P^{x\overset{t}{\to}y}_{\mathbf{B}}$ (and $E^{x\overset{t}{\to}y}_{\mathbf{B}}$) law (expectation) of the Brownian bridge $(\mathbf{B}_s)_{0\leq s\leq t}$ from x to y with lifetime t.

Liouville Brownian motion starting from $x \in \mathbb{T}$ formally defined by:

$$d\mathcal{B}_t = e^{-\frac{\gamma}{2}X(\mathbf{B}_t)}d\mathbf{B}_t$$

Liouville heat kernel: definition

Liouville Brownian motion starting from $x \in \mathbb{T}$:

$$\mathcal{B}_t = \mathbf{B}_{F(t)^{-1}}$$

where

$$F(t) = \int_0^t e^{\gamma X(\mathbf{B}_r) - \frac{\gamma^2}{2}\mathbb{E}^X[X^2(\mathbf{B}_r)]}\, dr.$$

Liouville heat kernel $\mathbf{p}_t^\gamma$ defined for all f by:

$$E^x_{\mathbf{B}}[f(\mathcal{B}_t)] = E^x_{\mathbf{B}}[f(\mathbf{B}_{F(t)^{-1}})] = \int_{\mathbb{T}} f(y)\mathbf{p}_t^\gamma(x,y)\, M_\gamma(dy), \;\; t > 0$$

Liouville heat kernel: representation and regularity

Consider the Hilbert-Schmidt operator:

$$T_\gamma : f \mapsto \int_{\mathbb{T}} G_\gamma(x,y) f(y) M_\gamma(dy)$$

with

$$G_\gamma(x,y) = G(x,y) - \frac{\int_{\mathbb{T}} G(z,y) M_\gamma(dz)}{M_\gamma(\mathbb{T})}$$

Let $(\lambda_{\gamma,n})_{n\geq 1}$ be the (increasing) eigenvalues of T_γ^{-1} associated to the eigenvectors $(\mathbf{e}_n^\gamma)_{n\geq 1}$. We have: $\sum_{n\geq 1} \frac{1}{\lambda_{\gamma,n}^2} < +\infty$.

Liouville heat kernel: representation and regularity

We have the following representation:

Theorem (Maillard, Rhodes, V., Zeitouni)

$$\mathbf{p}_t^\gamma(x,y) = \frac{1}{M_\gamma(\mathbb{T})} + \sum_{n\geq 1} e^{-\lambda_{\gamma,n} t} \mathbf{e}_n^\gamma(x) \mathbf{e}_n^\gamma(y).$$

Furthermore, it is of class $C^{\infty,0,0}(\mathbb{R}_+^* \times \mathbb{T}^2)$*. If* $\gamma < 2 - \sqrt{2}$*, it is even of class* $C^{\infty,1,1}(\mathbb{R}_+^* \times \mathbb{T}^2)$*.*

Speculations and heuristics

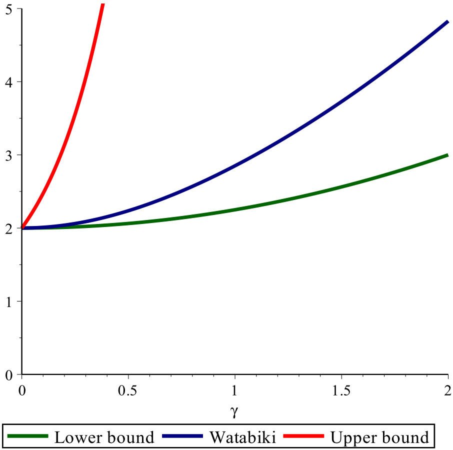
Watabiki (1993) conjectures that one can construct a metric space $(\mathbb{T}, \mathbf{d}_\gamma)$ which is locally monofractal with intrinsic Hausdorff dimension

$$d_H(\gamma) = 1 + \frac{\gamma^2}{4} + \sqrt{\left(1 + \frac{\gamma^2}{4}\right)^2 + \gamma^2}.$$

The literature on diffusion on fractals suggests that the heat kernel $\mathbf{p}_t^\gamma(x, y)$ then takes the following form for small t:

$$\mathbf{p}_t^\gamma(x, y) \asymp \frac{C}{t} exp\left(-C \frac{\mathbf{d}_\gamma(x, y)^{d_H(\gamma)/(d_H(\gamma)-1)}}{t^{\frac{1}{d_H(\gamma)-1}}}\right)$$

Summary of our bounds within these heuristics



The lower bound: fixed points

Theorem (Maillard, Rhodes, V., Zeitouni)

Fix $x \neq y$. For all $\eta > 0$, there exists some random variable $T_0 = T_0(x, y, \eta)$ such that for all $t \leq T_0$,

$$\mathbf{p}_t^{\gamma}(x, y) \geq \exp\Big(- t^{-\frac{1}{1+\gamma^2/4-\eta}} \Big), \quad \mathbb{P}^X\text{-a.s.}$$

The lower bound: typical points

Theorem (Maillard, Rhodes, V., Zeitouni)

Conditioned on the Gaussian field X, let x,y be sampled according to the measure $M_\gamma(\mathbb{T})^{-1}M_\gamma$. For all $\eta > 0$, there exists some random variable T_0, such that for all $t \leq T_0$,

$$\mathbf{p}_t^\gamma(x,y) \geq \exp\Big(- t^{-\frac{1}{\nu(\gamma)-\eta}}\Big), \quad \mathbb{P}^X\text{-a.s.},$$

where

$$\nu(\gamma) = \begin{cases} 1+\frac{\gamma^2}{4} & \gamma^2 \in [0,8/3] \\ 1+\gamma^2 - \frac{\gamma^2}{4}\left(1-\frac{\gamma^2}{4}\right)^{-1} & \gamma^2 \in (8/3,3] \\ 4-\gamma^2 & \gamma^2 \in (3,4). \end{cases}$$

Strategy of the proof for fixed points

Work with the resolvent which has the explicit representation:

$$\int_0^\infty e^{-\lambda t}\mathbf{p}_t^\gamma(x,y)dt = \int_0^\infty E_{\mathbf{B}}^{x\xrightarrow{t}y}\left[e^{-\lambda F(t)}\right] p_t(x,y)dt, \quad \lambda > 0.$$

Goal: give a lower bound of $E_{\mathbf{B}}^{x\xrightarrow{t}y}\left[e^{-\lambda F(t)}\right]$ (Brownian bridge in random environment)

Strategy: find an event A_t that costs $P_{\mathbf{B}}^{x\xrightarrow{t}y}(A_t) = e^{-c/t}$ such that $E_{\mathbf{B}}^{x\xrightarrow{t}y}\left[e^{-\lambda F(t)}|A_t\right]$ is big and use Jensen:

$$E_{\mathbf{B}}^{x\xrightarrow{t}y}\left[e^{-\lambda F(t)}|A_t\right] \geq e^{-\lambda E_{\mathbf{B}}^{x\xrightarrow{t}y}[F(t)|A_t]}$$

Strategy of the proof for fixed points

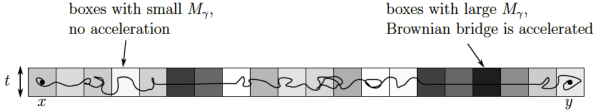


Figure: Strategy followed by the bridge in the boxes $(S_k^t)_{k\leq\frac{1}{t}}$ of side length t

Event A_t: the $(\mathbf{B}_s)_{s\leq t}$ stays in the corridor $[x,y]\times[-t,t]$ and is accelerated on the δ-thick boxes of M_γ.

Multifractal Analysis: $|\{k;\ M_\gamma(S_k^t)\approx t^{2+\delta\gamma-\gamma^2/2}\}|\approx t^{\delta^2/2-1}$.

Strategy of the proof for fixed points

Spending t_δ time in all δ-thick boxes costs $(e^{-c\frac{t^2}{t_\delta}})^{t^{\delta^2/2-1}} = e^{-c\frac{t^{1+\delta^2/2}}{t_\delta}}$. The cost is $e^{-c/t}$ for $t_\delta = t^{2+\delta^2/2}$.

Therefore, we will consider the event A_t that the bridge spends $t^{2+\delta^2/2}$ time in each δ-thick boxes of M_γ.

The contribution on $F(t)$ of the δ-thick boxes is then:

$$t^{\delta^2/2-1} t^{2+\delta^2/2} t^{\gamma^2/2-\delta\gamma} = t^{1+(\delta-\gamma/2)^2+\gamma^2/4} \leq t^{1+\gamma^2/4}$$

Conclusion: $F(t) \approx t^{1+\gamma^2/4}$ on the event A_t.

Strategy of the proof for fixed points

Going back to the Resolvent:

$$
\begin{aligned}
\int_0^\infty e^{-\lambda t}\mathbf{p}_t^\gamma(x,y)dt &= \int_0^\infty E_{\mathbf{B}}^{x\xrightarrow{t}y}\left[e^{-\lambda F(t)}\right] p_t(x,y)dt \\
&\geq \int_0^\infty E_{\mathbf{B}}^{x\xrightarrow{t}y}\left[e^{-\lambda F(t)}\middle|A_t\right] e^{-c/t}dt \\
&\geq \int_0^\infty e^{-\lambda E_{\mathbf{B}}^{x\xrightarrow{t}y}[F(t)|A_t]} e^{-c/t}dt \\
&\geq \int_0^\infty e^{-\lambda t^{1+\gamma^2/4}} e^{-c/t}dt \\
&\geq ce^{-c\lambda^{\frac{1}{2+\gamma^2/4}}}
\end{aligned}
$$

The upper bound: a useful general lemma

In the context of Liouville Brownian motion, we have the following lemma:

Lemma (Barlow, Grygorian)

Let $\beta > 1$, $\alpha > 0$ and $\tau_{B(y,r)}$ denotes the LBM exit time from the Euclidean ball $B(y,r)$. Assume that:

1) For all x, y and $t > 0$, we have $\mathbf{p}_t^\gamma(x,y) \leq C\Big(\frac{1}{t^\alpha} + 1\Big)$.

2) $\overline{\lim_{r\to 0}} \sup_{y\in\mathbb{T}} \mathbb{P}^y(\tau_{B(y,r)} \leq r^\beta) = 0$

Then, for all $t > 0$ and M_γ almost all $x, y \in \mathbb{T}$,

$$\mathbf{p}_t^\gamma(x,y) \leq C'\Big(\frac{1}{t^\alpha} + 1\Big) \exp\Big(- C'' \Big(\frac{d(x,y)}{t^{1/\beta}}\Big)^{\frac{\beta}{\beta-1}} \Big).$$

The upper bound

Set $\alpha = 2\Big(1 - \frac{\gamma}{2}\Big)^2$ and $\forall u > 0,\ \beta(u) = \Big(\frac{\gamma}{\sqrt{u}} + \sqrt{\frac{\gamma^2}{u} + 2 + \frac{\gamma^2}{2}}\ \Big)^2.$

Theorem (Maillard, Rhodes, V., Zeitouni)

For each $\delta > 0$, we set $\alpha_\delta = \alpha - \delta,\quad \beta_\delta = \beta(\alpha_\delta) + \delta$. Then, there exist two random constants $c_1 = c_1(X), c_2 = c_2(X) > 0$ such that

$$\forall x, y \in \mathbb{T}, t > 0, \quad \mathbf{p}_t^\gamma(x,y) \leq \frac{c_1}{t^{1+\delta}} \exp\Big(- c_2 \Big(\frac{d_{\mathbb{T}}(x,y)}{t^{1/\beta_\delta}} \Big)^{\frac{\beta_\delta}{\beta_\delta - 1}} \Big).$$

Remark

Using similar techniques, these bounds were improved recently by S. Andres, N. Kajino to $\beta = \frac{(\gamma+2)^2}{2}$.

The upper bound: strategy of the proof

By definition, for a fixed $y \in \mathbb{T}$

$$\tau_{B(y,r)} = F(T_{B(y,r)}) = \int_0^{T_{B(y,r)}} e^{\gamma X(\mathbf{B}_r) - \frac{\gamma^2}{2}\mathbb{E}^X[X^2(\mathbf{B}_r)]} \, dr.$$

where $T_{B(y,r)}$ is the exit time of a standard Brownian motion $\mathbf{B}$ starting from y.

One has for all $q < \frac{4}{\gamma^2}$

$$\mathbb{P}^X \mathbb{P}^y(\tau_{B(y,r)} \leq r^\beta) \leq r^{\beta q} \mathbb{E}^X \mathbb{E}^y [\frac{1}{F(T_{B(y,r)})^q}]$$

Using this relation on a fine grid and a union bound entails the bound β.

Other works on the Liouville heat kernel

- N. Berestycki, C. Garban, R. Rhodes, V.(2014): *KPZ formula derived from the Liouville heat kernel*
- S. Andres, N. Kajino (2014): *Continuity and estimates of the Liouville heat kernel with applications to spectral dimensions*
- M. Biskup, J. Ding : work in progress.

Liouville quantum gravity on the Riemann sphere

Why study Liouville quantum gravity on the Riemann sphere?

- Important conformal field theory (indexed by a continuum set of parameters) which is exactly solvable
- Scaling limit of random planar maps
- Link with $4d$-gauge theories

Liouville quantum gravity on the Riemann sphere

References:

- N. Seiberg (1990): *Notes on Quantum Liouville Theory and Quantum Gravity*
- Y. Nakayama (2004): *Liouville field theory: a decade after the revolution*
- D. Harlow, J. Maltz, E. Witten (2011): *Analytic continuation of Liouville theory*

Liouville quantum gravity on the Riemann sphere

Consider the following partition function on the sphere (Polyakov 1981)

$$Z = \int e^{-S_L(X,\hat{g})} DX$$

where S_L is the Liouville action:

$$S_L(X,\hat{g}) := \frac{1}{4\pi} \int_{\mathbb{R}^2} \left(|\partial_{\hat{g}} X|^2(x) + Q R_{\hat{g}}(x) X(x) + 4\pi\mu e^{\gamma X(x)} \right) \hat{g}(x) dx$$

and $\hat{g}$ some reference metric on the sphere.
Goal: construct a CFT independent of the reference metric (within the same conformal equivalence class) with action given by S_L.
Here, we will choose $\hat{g}(x) = \frac{4}{(1+|x|^2)^2}$.

Liouville quantum gravity on the Riemann sphere: the Gaussian Free Field

We denote:

- $\triangle_{\hat{g}}$ Laplacian
- G Green function with vanishing mean,
- GFF X: $\mathbb{E}[X(x)X(y)] = G(x,y)$.
- Liouville field: $X(x) + \frac{Q}{2} \ln \hat{g}(x)$
- Regularized GFF: $X_\epsilon(x) = \frac{1}{2\pi\epsilon} \int_0^{2\pi} X(x + \epsilon e^{i\theta}) d\theta$
- Vertex operator: $V_{\alpha,\epsilon}(x) := \epsilon^{\alpha^2/2}\, e^{\alpha(X_\epsilon(x) + \frac{Q}{2} \ln \hat{g}(x))}$

Liouville quantum gravity on the Riemann sphere: the n-point correlation function

Goal: construct a CFT on the sphere.

Problem: if ψ Mobius transform, $X \circ \psi \overset{(Law)}{\neq} X$.

Liouville quantum gravity on the Riemann sphere: the n-point correlation function

Goal: construct a CFT on the sphere.

Problem: if ψ Mobius transform, $X \circ \psi \overset{(Law)}{\neq} X$.

In order to ensure conformal invariance, we need to integrate with respect to the Lebesgue measure. Hence, we get the following definition:

$$
\begin{aligned}
& C((\alpha_i),(z_i),\mu,F(.)) \\
&= \lim_{\epsilon \to 0} \int_{\mathbb{R}} \mathbb{E}\Big[\left(\prod_i \epsilon^{\frac{\alpha_i^2}{2}} e^{\alpha_i (c + X_\epsilon + \frac{Q}{2} \ln \hat{g})(z_i)} \right) e^{-\frac{Q}{4\pi} \int_{\mathbb{R}^2} 2(c + X(x) + \frac{Q}{2} \ln \hat{g}(x)) \, \hat{g}(x) dx} \\
& F(.) \ \exp\left(-\mu e^{\gamma c} \epsilon^{\frac{\gamma^2}{2}} \int_{\mathbb{R}^2} e^{\gamma (X_\epsilon(x) + Q/2 \ln \hat{g}(x))} \, dx \right) \Big] \, dc.
\end{aligned}
$$

Liouville quantum gravity on the Riemann sphere

$$
\begin{aligned}
&C((\alpha_i),(z_i),\mu,F(.))\\
&=\lim_{\epsilon\to 0}\int_{\mathbb{R}}\mathbb{E}\Big[\prod_i \epsilon^{\frac{\alpha_i^2}{2}} e^{\alpha_i(c+X_\epsilon+\frac{Q}{2}\ln\hat{g})(z_i)} e^{-\frac{Q}{4\pi}\int_{\mathbb{R}^2} 2(c+X(x)+\frac{Q}{2}\ln\hat{g}(x))\,\hat{g}(x)dx}\\
&\quad F(.)\ \exp\Big(-\mu e^{\gamma c}\epsilon^{\frac{\gamma^2}{2}}\int_{\mathbb{R}^2} e^{\gamma(X_\epsilon(x)+Q/2\ln\hat{g}(x))}\,dx\Big)\Big]\,dc.
\end{aligned}
$$

We set $C((\alpha_i),(z_i),\mu) = C((\alpha_i),(z_i),\mu,F(.)=1)$ and the probability

$$
\mathbb{E}^{(z_i),(\alpha_i),\mu}[F(.)] = \frac{C((\alpha_i),(z_i),\mu,F(.))}{C((\alpha_i),(z_i),\mu)}.
$$

Liouville quantum gravity on the Riemann sphere: existence of the n-point correlation

Theorem (David, Kupiainen, Rhodes, V.)

If $\sum_i \alpha_i > 2Q$ and $\alpha_i < Q$ (Seiberg bound), then

$$
\begin{aligned}
& C((\alpha_i),(z_i),\mu) \\
& = \Big(\prod_i \hat{g}(z_i)^{-\frac{\alpha_i^2}{4}+\frac{Q}{2}\alpha_i}\Big) e^{\sum_{i\neq j}\alpha_i\alpha_j G(z_i,z_j)} e^{C(\hat{g})} \frac{\mu^{\frac{2Q-\sum_i \alpha_i}{\gamma}}}{\gamma} \\
& \times \Gamma\Big(\gamma^{-1}(\sum_i \alpha_i - 2Q)\Big) \mathbb{E}\Big[\frac{1}{Z_{(z_i)}(\mathbb{R}^2)^{\frac{\sum_i \alpha_i - 2Q}{\gamma}}}\Big]
\end{aligned}
$$

where Γ is the standard gamma function, $C(\hat{g})$ a global constant and

$$
Z_{(z_i)}(dx) = e^{\gamma X(x) - \frac{\gamma^2}{2}\mathbb{E}[X(x)^2] + \gamma \sum_i \alpha_i G(x,z_i)} dx
$$

Liouville quantum gravity on the Riemann sphere: conformal invariance of the n-point correlation

Theorem (David, Kupiainen, Rhodes, V.)

If ψ is a Mobius transform then

$$C((\alpha_i),(\psi(z_i)),\mu)=\prod_i |\psi'(z_i)|^{-2\Delta_{\alpha_i}}C((\alpha_i),(z_i),\mu)$$

where Δ_{α_i} are the conformal weights: $\Delta_{\alpha_i}=\frac{\alpha_i}{2}(Q-\frac{\alpha_i}{2})$.

Proof: use definition of $C((\alpha_i),(z_i),\mu)$ as a limit and then:
Girsanov+ computations involving change of metrics +
$X\circ\psi-\frac{1}{4\pi}\int_{\mathbb{R}^2}X\circ\psi(x)\hat{g}(x)dx \overset{(Law)}{=} X$

Liouville quantum gravity on the Riemann sphere: the Liouville measure

The Liouville measure $Z_L(dx) = \lim_\epsilon e^{\gamma(X_\epsilon(x) + \frac{Q}{2}\ln \hat{g}(x))}dx$ is conformally invariant (with respect to Mobius) and its total mass has a Γ distribution

$$\mathbb{E}^{(z_i),(\alpha_i),\mu}[F(Z_L(\mathbb{R}^2))] = \frac{\mu^{\frac{2Q-\sum_i \alpha_i}{\gamma}}}{\Gamma(\frac{\sum_i \alpha_i - 2Q}{\gamma})} \int_0^\infty F(v) v^{\frac{\sum_i \alpha_i - 2Q}{\gamma} - 1} e^{-\mu v} dv$$

Conditioning on the volume, we get the distribution

$$\mathbb{E}^{(z_i),(\alpha_i),\mu}[Z_L(dx) | Z_L(\mathbb{R}^2) = A] = \frac{\mathbb{E}[F\left(A\frac{Z_{(z_i)}(dx)}{Z_{(z_i)}(\mathbb{R}^2)}\right) Z_{(z_i)}(\mathbb{R}^2)^{-\frac{\sum_i \alpha_i - 2Q}{\gamma}}]}{\mathbb{E}[Z_{(z_i)}(\mathbb{R}^2)^{-\frac{\sum_i \alpha_i - 2Q}{\gamma}}]}$$

Liouville quantum gravity on the Riemann sphere: perspectives

Perspectives:

- Compute the semi-classical limit of your system, i.e. $\gamma \to 0$ (Liouville equation on the sphere with conical singularities)
- Give explicit expressions for the 3 point correlation function (conjectured to be the 3 point correlations of numerous $2d$-statistical physics systems at criticality)