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   - Construction
   - Basic properties
We equip the two dimensional torus $\mathbb{T}$ with:

- $d_\mathbb{T}$ standard volume distance and $dx$ volume form
- $\Delta$ the Laplace-Beltrami operator on $\mathbb{T}$
- $p_t(x, y)$ the standard heat kernel of the Brownian motion $B$ on $\mathbb{T}$

Recall that:

$$p_t(x, y) = \frac{1}{|\mathbb{T}|} + \sum_{n \geq 1} e^{-\lambda_n t} e_n(x)e_n(y)$$

where $(\lambda_n)_{n \geq 1}$ (increasing) eigenvalues and $(e_n)_{n \geq 1}$ (normalized) eigenvectors:

$$-\Delta e_n = 2\pi \lambda_n e_n, \quad \int_{\mathbb{T}} e_n(x)dx = 0.$$
Log-correlated field $X$

Notations:

- $G$ standard Green function of the Laplacian $\Delta$:
  \[
  G(x, y) = \sum_{n \geq 1} \frac{1}{\lambda_n} e_n(x) e_n(y)
  \]

- $X$ GFF on $\mathbb{T}$ under $\mathbb{P}^X$ (expectation $\mathbb{E}^X$):
  \[
  \mathbb{E}^X[X(x)X(y)] = G(x, y) = \ln + \frac{1}{d_{\mathbb{T}}(x, y)} + g(x, y)
  \]
Gaussian multiplicative chaos (Liouville measure)

Gaussian multiplicative chaos associated to $X$:

$$M_\gamma(dx) = e^{\gamma X(x) - \frac{\gamma^2}{2} \mathbb{E}[X(x)^2]} \, dx.$$ 

**Theorem (Kahane, 1985)**

$M_\gamma$ can be defined by regularizing the field $X$ and a limit procedure. $M_\gamma \neq 0$ if and only if $\gamma < 2$. If $\gamma < 2$, the measure $M_\gamma$ "lives" almost surely on a set of Hausdorff dimension $2 - \frac{\gamma^2}{2}$ (the set of thick points).
Liouville Brownian motion

Framework:

- Standard Brownian motion $\mathcal{B} = (\mathcal{B}_t)_{t \geq 0}$ on $\mathbb{T}$
- $P_{\mathcal{B}}^{\mathcal{X}}$ (and $E_{\mathcal{B}}^{\mathcal{X}}$) probability (expectation) of $\mathcal{B}$ starting from $x$.
- $P_{\mathcal{B}}^{x \xrightarrow{t} y}$ (and $E_{\mathcal{B}}^{x \xrightarrow{t} y}$) law (expectation) of the Brownian bridge $(\mathcal{B}_s)_{0 \leq s \leq t}$ from $x$ to $y$ with lifetime $t$.

Liouville Brownian motion starting from $x \in \mathbb{T}$ formally defined by:

$$d\mathcal{B}_t = e^{-\frac{\gamma}{2}X(\mathcal{B}_t)}d\mathcal{B}_t$$
Liouville heat kernel: definition

Liouville Brownian motion starting from $x \in \mathbb{T}$:

$$ B_t = B_{F(t)}^{-1} $$

where

$$ F(t) = \int_0^t e^{\gamma X(B_r) - \frac{\gamma^2}{2} E^X[X^2(B_r)]} \, dr. $$

Liouville heat kernel $p_t^\gamma$ defined for all $f$ by:

$$ E^x_B[f(B_t)] = E^x_B[f(B_{F(t)}^{-1})] = \int_{\mathbb{T}} f(y) p_t^\gamma(x, y) M_\gamma(dy), \ t > 0 $$
Consider the Hilbert-Schmidt operator:

$$T_\gamma : f \mapsto \int_{\mathbb{T}} G_\gamma(x, y) f(y) M_\gamma(dy)$$

with

$$G_\gamma(x, y) = G(x, y) - \int_{\mathbb{T}} G(z, y) M_\gamma(dz)$$

Let $\lambda_{\gamma,n} \geq 1$ be the (increasing) eigenvalues of $T^{-1}_\gamma$ associated to the eigenvectors $(e_\gamma^n)_{n \geq 1}$. We have:

$$\sum_{n \geq 1} \frac{1}{\lambda_{\gamma,n}^2} < +\infty.$$
Liouville heat kernel: representation and regularity

We have the following representation:

**Theorem (Maillard, Rhodes, V., Zeitouni)**

\[ p_t^\gamma(x, y) = \frac{1}{M_\gamma(T)} + \sum_{n \geq 1} e^{-\lambda_\gamma n t} e_\gamma^n(x) e_\gamma^n(y). \]

Furthermore, it is of class $C^{\infty, 0, 0}(\mathbb{R}^*_+ \times T^2)$. If $\gamma < 2 - \sqrt{2}$, it is even of class $C^{\infty, 1, 1}(\mathbb{R}^*_+ \times T^2)$. 
Watabiki (1993) conjectures that one can construct a metric space $(\mathbb{T}, d_\gamma)$ which is locally monofractal with intrinsic Hausdorff dimension

$$d_H(\gamma) = 1 + \frac{\gamma^2}{4} + \sqrt{(1 + \frac{\gamma^2}{4})^2 + \gamma^2}.$$  

The literature on diffusion on fractals suggests that the heat kernel $p_t^\gamma(x, y)$ then takes the following form for small $t$:

$$p_t^\gamma(x, y) \asymp \frac{C}{t} \exp \left( -C \frac{d_\gamma(x, y)^{d_H(\gamma)/(d_H(\gamma)-1)}}{t^{\frac{1}{d_H(\gamma)-1}}} \right)$$
Summary of our bounds within these heuristics
The lower bound: fixed points

**Theorem (Maillard, Rhodes, V., Zeitouni)**

Fix \( x \neq y \). For all \( \eta > 0 \), there exists some random variable \( T_0 = T_0(x, y, \eta) \) such that for all \( t \leq T_0 \),

\[
p_t^{\gamma}(x, y) \geq \exp \left( - t^{\frac{1}{1+\gamma^2/4-\eta}} \right), \quad \mathbb{P}^X \text{-a.s.}
\]
The lower bound: typical points

**Theorem (Maillard, Rhodes, V., Zeitouni)**

Conditioned on the Gaussian field $X$, let $x, y$ be sampled according to the measure $M_\gamma(T)^{-1}M_\gamma$. For all $\eta > 0$, there exists some random variable $T_0$, such that for all $t \leq T_0$,

$$p_\gamma^t(x, y) \geq \exp \left( - t \frac{1}{\nu(\gamma) - \eta} \right), \quad \mathbb{P}^X-\text{a.s.},$$

where

$$\nu(\gamma) = \begin{cases} 
1 + \gamma^2 & \gamma^2 \in [0, 8/3] \\
1 + \gamma^2 - \gamma^2 \left(1 - \frac{\gamma^2}{4}\right)^{-1} & \gamma^2 \in (8/3, 3] \\
4 - \gamma^2 & \gamma^2 \in (3, 4).
\end{cases}$$
Strategy of the proof for fixed points

Work with the resolvent which has the explicit representation:

\[ \int_0^\infty e^{-\lambda t} p_t(\gamma)(x, y) dt = \int_0^\infty E_{B}^{x \rightarrow y} \left[ e^{-\lambda F(t)} \right] p_t(x, y) dt, \quad \lambda > 0. \]

Goal: give a lower bound of \( E_{B}^{x \rightarrow y} \left[ e^{-\lambda F(t)} \right] \) (Brownian bridge in random environment)

Strategy: find an event \( A_t \) that costs \( P_{B}^{x \rightarrow y}(A_t) = e^{-c/t} \) such that \( E_{B}^{x \rightarrow y} \left[ e^{-\lambda F(t)} | A_t \right] \) is big and use Jensen:

\[ E_{B}^{x \rightarrow y} \left[ e^{-\lambda F(t)} | A_t \right] \geq e^{-\lambda E_{B}^{x \rightarrow y}[F(t) | A_t]} \]
Strategy of the proof for fixed points

Event $A_t$: the $(B_s)_{s \leq t}$ stays in the corridor $[x, y] \times [-t, t]$ and is accelerated on the $\delta$-thick boxes of $M_\gamma$. 

Multifractal Analysis: $|\{k; M_\gamma(S_k^t) \approx t^{2+\delta\gamma-\gamma^2/2}\}| \approx t^{\delta^2/2-1}$.
Spending $t_\delta$ time in all $\delta$-thick boxes costs

$$\left(e^{-c\frac{t^2}{t_\delta}}\right)t^{\delta^2/2-1} = e^{-c\frac{t^{1+\delta^2/2}}{t_\delta}}.$$ 

The cost is $e^{-c/t}$ for $t_\delta = t^{2+\delta^2/2}$. 

Therefore, we will consider the event $A_t$ that the bridge spends $t^{2+\delta^2/2}$ time in each $\delta$-thick boxes of $M_\gamma$.

The contribution on $F(t)$ of the $\delta$-thick boxes is then:

$$t^{\delta^2/2-1} t^{2+\delta^2/2} t^{\gamma^2/2-\delta\gamma} = t^{1+(\delta-\gamma/2)^2+\gamma^2/4} \leq t^{1+\gamma^2/4}$$

Conclusion: $F(t) \approx t^{1+\gamma^2/4}$ on the event $A_t$. 
Strategy of the proof for fixed points

Going back to the Resolvent:

\[
\int_0^\infty e^{-\lambda t} p_t^\gamma(x, y) dt = \int_0^\infty E_{B}^{x \rightarrow y} \left[ e^{-\lambda F(t)} \right] p_t(x, y) dt \\
\geq \int_0^\infty E_{B}^{x \rightarrow y} \left[ e^{-\lambda F(t)} | A_t \right] e^{-c/t} dt \\
\geq \int_0^\infty e^{-\lambda E_{B}^{x \rightarrow y} [F(t)|A_t]} e^{-c/t} dt \\
\geq \int_0^\infty e^{-\lambda t^{1+\gamma^2/4}} e^{-c/t} dt \\
\geq c e^{-c \lambda^{2+\gamma^2/4}}
\]
The upper bound: a useful general lemma

In the context of Liouville Brownian motion, we have the following lemma:

**Lemma (Barlow,Grygorian)**

Let $\beta > 1$, $\alpha > 0$ and $\tau_{B(y,r)}$ denotes the LBM exit time from the Euclidean ball $B(y, r)$. Assume that:

1) For all $x, y$ and $t > 0$, we have $p^\gamma_t(x, y) \leq C\left(\frac{1}{t^\alpha} + 1\right)$.

2) $\lim_{r \to 0} \sup_{y \in \mathbb{T}} \mathbb{P}^y(\tau_{B(y,r)} \leq r^\beta) = 0$

Then, for all $t > 0$ and $M_\gamma$ almost all $x, y \in \mathbb{T}$,

$$p^\gamma_t(x, y) \leq C'\left(\frac{1}{t^\alpha} + 1\right) \exp\left(-C''\left(\frac{d(x, y)}{t^{1/\beta}}\right)^{\frac{\beta}{\beta-1}}\right).$$
The upper bound

Set \( \alpha = 2 \left( 1 - \frac{\gamma}{2} \right)^2 \) and \( \forall u > 0, \beta(u) = \left( \frac{\gamma}{\sqrt{u}} + \sqrt{\frac{\gamma^2}{u} + 2 + \frac{\gamma^2}{2}} \right)^2 \).

**Theorem (Maillard, Rhodes, V., Zeitouni)**

For each \( \delta > 0 \), we set \( \alpha_\delta = \alpha - \delta, \beta_\delta = \beta(\alpha_\delta) + \delta \). Then, there exist two random constants \( c_1 = c_1(X), c_2 = c_2(X) > 0 \) such that

\[
\forall x, y \in \mathcal{T}, t > 0, \quad p^\gamma_t(x, y) \leq \frac{c_1}{t^{1+\delta}} \exp \left( - c_2 \left( \frac{d_\mathcal{T}(x, y)}{t^{1/\beta_\delta}} \right)^{\beta_\delta - 1} \right).
\]

**Remark**

*Using similar techniques, these bounds were improved recently by S. Andres, N. Kajino to* \( \beta = \frac{\gamma + 2}{2} \).
By definition, for a fixed $y \in \mathbb{T}$

$$
\tau_{B(y,r)} = F(T_{B(y,r)}) = \int_0^{T_{B(y,r)}} e^{\gamma X(B_r) - \frac{\gamma^2}{2} E^X[X^2(B_r)]} \, dr.
$$

where $T_{B(y,r)}$ is the exit time of a standard Brownian motion $B$ starting from $y$.

One has for all $q < \frac{4}{\gamma^2}$

$$
\mathbb{P}^X \mathbb{P}^y(\tau_{B(y,r)} \leq r^\beta) \leq r^{\beta q} E^X E^y \left[ \frac{1}{F(T_{B(y,r)})} \right]^q
$$

Using this relation on a fine grid and a union bound entails the bound $\beta$. 
Other works on the Liouville heat kernel

- N. Berestycki, C. Garban, R. Rhodes, V. (2014): *KPZ formula derived from the Liouville heat kernel*
- S. Andres, N. Kajino (2014): *Continuity and estimates of the Liouville heat kernel with applications to spectral dimensions*
- M. Biskup, J. Ding: work in progress.
Liouville quantum gravity on the Riemann sphere

Why study Liouville quantum gravity on the Riemann sphere?

- Important conformal field theory (indexed by a continuum set of parameters) which is exactly solvable
- Scaling limit of random planar maps
- Link with 4d-gauge theories
Liouville quantum gravity on the Riemann sphere

References:

- N. Seiberg (1990): *Notes on Quantum Liouville Theory and Quantum Gravity*
- Y. Nakayama (2004): *Liouville field theory: a decade after the revolution*
- D. Harlow, J. Maltz, E. Witten (2011): *Analytic continuation of Liouville theory*
Consider the following partition function on the sphere (Polyakov 1981)

\[ Z = \int e^{-S_L(X, \hat{g})} DX \]

where \( S_L \) is the Liouville action:

\[ S_L(X, \hat{g}) := \frac{1}{4\pi} \int_{\mathbb{R}^2} \left( |\partial \hat{g} X|^2(x) + Q R \hat{g}(x) X(x) + 4\pi \mu e^{\gamma X(x)} \right) \hat{g}(x) dx \]

and \( \hat{g} \) some reference metric on the sphere.

Goal: construct a CFT independent of the reference metric (within the same conformal equivalence class) with action given by \( S_L \).

Here, we will choose \( \hat{g}(x) = \frac{4}{(1+|x|^2)^2} \).
We denote:

- $\triangle \hat{g}$ Laplacian
- $G$ Green function with vanishing mean,
- GFF $X$: $\mathbb{E}[X(x)X(y)] = G(x, y)$.
- Liouville field: $X(x) + \frac{Q}{2} \ln \hat{g}(x)$
- Regularized GFF: $X_\epsilon(x) = \frac{1}{2\pi\epsilon} \int_0^{2\pi} X(x + \epsilon e^{i\theta}) d\theta$
- Vertex operator: $V_{\alpha, \epsilon}(x) := \epsilon^{\alpha^2/2} e^{\alpha(X_\epsilon(x) + \frac{Q}{2} \ln \hat{g}(x))}$
Liouville quantum gravity on the Riemann sphere: the n-point correlation function

Goal: construct a CFT on the sphere.

Problem: if $\psi$ Mobius transform, $X \circ \psi^{(Law)} \neq X$. 
Liouville quantum gravity on the Riemann sphere: the n-point correlation function

Goal: construct a CFT on the sphere.

Problem: if $\psi$ Mobius transform, $X \circ \psi \neq X$.

In order to ensure conformal invariance, we need to integrate with respect to the Lebesgue measure. Hence, we get the following definition:

$$C((\alpha_i), (z_i), \mu, F(.)) = \lim_{\epsilon \to 0} \int \mathbb{E} \left[ \left( \prod_i \epsilon \frac{\alpha_i^2}{2} e^{\alpha_i(c+X\epsilon + \frac{Q}{2} \ln \hat{g})(z_i)} \right) e^{-\frac{Q}{4\pi} \int_{\mathbb{R}^2} 2(c+X(x)+\frac{Q}{2} \ln \hat{g}(x)) \hat{g}(x) dx} \right] \ F(.) \ exp \left( -\mu e^{\gamma c} \epsilon \frac{\gamma^2}{2} \int_{\mathbb{R}^2} e^{\gamma (X\epsilon(x)+Q/2 \ln \hat{g}(x))} dx \right) \ dc.$$
Liouville quantum gravity on the Riemann sphere

\[
C((\alpha_i), (z_i), \mu, F(.))
\]

\[
= \lim_{\epsilon \to 0} \int_{\mathbb{R}} \mathbb{E} \left[ \prod_i \epsilon^{\alpha_i^2/2} e^{\alpha_i \left( c + X_\epsilon + \frac{Q}{2} \ln \hat{g}(z_i) \right)} e^{-\frac{Q}{4\pi} \int_{\mathbb{R}^2} 2(c + X(x) + \frac{Q}{2} \ln \hat{g}(x)) \hat{g}(x) dx} \right] dc.
\]

We set \(C((\alpha_i), (z_i), \mu) = C((\alpha_i), (z_i), \mu, F(.)) = 1\) and the probability

\[
\mathbb{E}^{(z_i), (\alpha_i), \mu}[F(.)] = \frac{C((\alpha_i), (z_i), \mu, F(.))}{C((\alpha_i), (z_i), \mu)}.
\]
Liouville quantum gravity on the Riemann sphere: existence of the n-point correlation

**Theorem (David, Kupiainen, Rhodes, V.)**

If $\sum_i \alpha_i > 2Q$ and $\alpha_i < Q$ (Seiberg bound), then

$$C((\alpha_i), (z_i), \mu)$$

$$= \left( \prod_i \hat{g}(z_i) - \frac{\alpha_i^2}{4} + \frac{Q}{2} \alpha_i \right) e^{\sum_{i \neq j} \alpha_i \alpha_j G(z_i, z_j)} e^{C(\hat{g}) \mu} \frac{2Q - \sum_i \alpha_i}{\gamma}$$

$$\times \Gamma \left( \frac{\gamma}{\gamma - 1} \left( \sum_i \alpha_i - 2Q \right) \right) \mathbb{E} \left[ \frac{1}{Z(z_i)(\mathbb{R}^2)} \frac{\sum_i \alpha_i - 2Q}{\gamma} \right]$$

where $\Gamma$ is the standard gamma function, $C(\hat{g})$ a global constant and

$$Z(z_i)(dx) = e^{\gamma X(x) - \frac{\gamma^2}{2} \mathbb{E}[X(x)^2] + \gamma \sum_i \alpha_i G(x, z_i)} \, dx$$
Theorem (David, Kupiainen, Rhodes, V.)

If $\psi$ is a Mobius transform then

$$C((\alpha_i), (\psi(z_i)), \mu) = \prod_{i} |\psi'(z_i)|^{-2\Delta_{\alpha_i}} C((\alpha_i), (z_i), \mu)$$

where $\Delta_{\alpha_i}$ are the conformal weights: $\Delta_{\alpha_i} = \frac{\alpha_i}{2} (Q - \frac{\alpha_i}{2})$.

Proof: use definition of $C((\alpha_i), (z_i), \mu)$ as a limit and then:

Girsanov+ computations involving change of metrics +

$X \circ \psi - \frac{1}{4\pi} \int_{\mathbb{R}^2} X \circ \psi(x) \hat{g}(x) dx \overset{(Law)}{=} X$
Liouville quantum gravity on the Riemann sphere: the Liouville measure

The Liouville measure $Z_L(dx) = \lim_{\epsilon} e^{\gamma(X_\epsilon(x) + \frac{Q}{2} \ln \hat{g}(x))} dx$ is conformally invariant (with respect to Mobius) and its total mass has a $\Gamma$ distribution

$$E(z_i, (\alpha_i), \mu) [F(Z_L(\mathbb{R}^2))] = \int_0^\infty F(v) \frac{\sum_i \frac{\alpha_i - 2Q}{\gamma} }{\Gamma\left(\frac{\sum_i \alpha_i - 2Q}{\gamma}\right)} \frac{\sum_i \frac{\alpha_i - 2Q}{\gamma} }{\gamma} - 1 e^{-\mu v} dv$$

Conditioning on the volume, we get the distribution

$$E(z_i, (\alpha_i), \mu) [Z_L(dx) | Z_L(\mathbb{R}^2) = A] = \frac{E[F \left( A \frac{Z(z_i)(dx)}{Z(z_i)(\mathbb{R}^2)} \right) Z(z_i)(\mathbb{R}^2) - \frac{\sum_i \alpha_i - 2Q}{\gamma} ]}{E[Z(z_i)(\mathbb{R}^2) - \frac{\sum_i \alpha_i - 2Q}{\gamma} ]}$$
Perspectives:

- Compute the semi-classical limit of your system, i.e. $\gamma \rightarrow 0$ (Liouville equation on the sphere with conical singularities)
- Give explicit expressions for the 3 point correlation function (conjectured to be the 3 point correlations of numerous 2d-statistical physics systems at criticality)