The Three-dimensional Euler Equations: Recent Advances Through Examples

Edriss S. Titi

Weizmann Institute of Science and
The University of California - Irvine

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   - Euler equations

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Euler equations

We consider the Euler Equations of inviscid incompressible fluid in $\Omega = \mathbb{R}^3$, the whole space, or $\Omega = (\mathbb{R}/\mathbb{Z})^3$, the three dimensional torus.

$$\begin{aligned} u_t + (u \cdot \nabla)u + \nabla p &= 0, \quad x \in \Omega, \quad t > 0 \\ \nabla \cdot u &= 0, \\ u(x,0) &= u_0(x). \end{aligned}$$

where the velocity field $u = (u_1, u_2, u_3)$ and pressure p are unknowns.

Vorticity formulation

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the velocity can be recovered from the vorticity via the Biot-Savart law in $\mathbb{R}^3$

$$u(x,t) = \frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{(x-y) \wedge \omega(y)}{|x-y|^3} dy \,.$$

Classical well-posedness results

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- Question: Does there exist a regular solution (say in $C^{1,\alpha}$) of the $3d$ Euler equations that becomes singular in a finite time (blows up problem)?

A trivial blowup criterion

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$$\frac{\partial \theta}{\partial t} + u \cdot \nabla \theta = 0$$

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$$\xi = \frac{\omega}{|\omega|}$$

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Consider the Euler equations in a smooth bounded domain Ω with boundary condition $u \cdot \vec{n}|_{\partial\Omega} = 0$.

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Lack of uniqueness of weak solutions

Theorem DeLellis - Szekelyhidi

There exist a set of initial data $u_0 \in L^2(\Omega)$ *(not explicitly constructed) for which the Cauchy problem has, for the same initial data, an infinite family of weak solutions of the three-dimensional Euler equations: a residual set in the space* $C(\mathbb{R}_t; L^2_{\text{weak}}(\Omega))$.

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- Remark: Earlier results were established by Shnirelman and by V. Sheffer.

Theorem Wiedemann (2011).

There exists a family (non-uniqueness) of weak solutions to the Cauchy problem of the 3D Euler.

Ruling out Principle of wild weak solutions

Ruling out Principle

Any wild weak solution of Euler equations that cannot be achieved as a vanishing viscosity limit of the Navier-Stokes equations should be ruled out.

Does 2D Flow Remain 2D?

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Let u_0 be a function of (x, y), then the Leray-Hopf weak solution of the 3D Navier-Stokes remains a function of only (x, y).

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- Remark: Ruling out principle: all the wild weak solutions of Euler equations that do not obey the two-dimensional symmetry of the initial data should be ruled out. Because they cannot be obtained as vanishing viscosity limit of Navier-Stokes solutions.

Shear Flow

We consider in the whole space $\mathbb{R}^3$ or on the periodic box $(\mathbb{R}/\mathbb{Z})^3$:

The shear flow

$$u(x,t) = (u_1(x_2), 0, u_3(x_1 - tu_1(x_2)).$$

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- For $u_1, u_3 \in C^1$, the above shear flow is a classical solution of the Euler equations with pressure $p = 0$.
- Moreover, in the case of the periodic box $(\mathbb{R}/\mathbb{Z})^3$ the above shear flow conserves the energy.

Some history

- This example of shear flow was used by DiPerna and Majda (1987) to show that a weak limit of oscillating smooth solutions of the three-dimensional Euler equations may not be a solution of Euler equations.

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Idea of the proof

$$\partial_{x_2} u_3(x_1 - tu_1(x_2)) = -t\partial_{x_2} u_1(x_2)\partial_{x_1} u_3(x_1 - tu_1(x_2))$$

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If $\partial_{x_2} u_1, \partial_{x_1} u_3 \in L^p$ this does not imply that $\partial_{x_2} u_3 \in L^p$.

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$$\partial_t \overline{u} + \nabla \cdot (\overline{u} \otimes \overline{u}) + \nabla \cdot RT(u_\epsilon) + \nabla \overline{p} = 0 \ \text{ in } \Omega\,,$$

$$RT(u_\epsilon)(x,t) = \lim_{\epsilon \to 0} ((u_\epsilon - \overline{u}) \otimes (u_\epsilon - \overline{u}))$$

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Original DiPerna–Majda example: Sequence of weak solutions with energy estimate:

$$\nabla \cdot \overline{u} = 0 \ \text{ in } \Omega\,, \quad \overline{u} \cdot \vec{n} = 0 \ \text{ on } \partial\Omega\,,$$

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Shear flow is a weak solution that conserves energy

Theorem Bardos-Titi

(i) Let $u_1, u_3 \in L^2_{\text{loc}}(\mathbb{R})$ *then the shear flow*

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(ii) Let $u_1, u_3 \in L^2(\mathbb{R}/\mathbb{Z})$ *then the shear flow defined above is a weak solution of the Euler equations, in the sense of distribtions, in* $\Omega = (\mathbb{R}/\mathbb{Z})^3$. *Furthermore, in this case the energy of this solution is constant.*

Theorem *(i) For* $u_1(x), u_3(x) \in C^{1,\alpha}$*, with* $\alpha \in (0,1]$*, the shear flow solution*

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(iii) There exist shear flow solutions of the above form which, for $t = 0$, belong to $C^{0,\alpha}$, for some $\alpha \in (0,1)$, and for $t \neq 0$, they do not belong to $C^{0,\beta}$ for any $\beta > \alpha^2$.

Proof of the Theorem

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Next we show that this result is sharp.

Let us introduce two periodic functions $u_1(s)$ and $u_3(s)$ which near the point $s = 0$ coincide with $|s|^\alpha$ then the for t given and x_1 and x_2 small enough $u_3(x_1 - tu_3(x_2))$ coincides with

$$|x_1 - t|x_2|^\alpha|^\alpha .$$

For $(x_1, x_2, x_3) = (0, x_2, x_3)$ one has

$$u_3(x_1 - tu_3(x_2)) = |t|^\alpha |x_2|^{\alpha^2}$$

and the conclusion follows.

Other spaces and optimal spaces

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$$C^{1,\alpha} = B^{1+\alpha}_{\infty,\infty} \subset B^1_{\infty,1} \subset C^1 \subset F^1_{\infty,2} \subset B^1_{\infty,\infty} \subset B^{\alpha}_{\infty,\infty} = C^{0,\alpha}\,.$$

Besov and Trieble-Lizorkin spaces

Theorem *The $3d$ Euler equation is well posed in $B^1_{\infty,1}$* (Pak and Park). It is not well posed in $B^1_{\infty,\infty}$ or in the Triebel-Lizorkin space $\subset F^1_{\infty,2}$

Proof

$B^1_{\infty,\infty}$ is the Zygmund class, i.e. bounded functions with

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Same proof for $\subset F^1_{\infty,2}$. More delicate: Construction of a log Lipschitz function in this space.

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to provide an explicit example for an analytic solutions whose radius of analyticity is shrinking with the rate $\frac{1}{t}$.

Theorem[Bardos–Titi–Wiedemann, *C.R. Acad. Sci.– Paris*, (2012)].
Let $v_0(x) = (v_1(x_2), 0, v_3(x_1, x_2))$*, where we assume* $v_1 \in L^2(\mathbb{T})$ *and* $v_3 \in L^2(\mathbb{T}^2)$*.*

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$$\begin{aligned} \Sigma(t) = \{(x_1, x_2, x_3) | \, x_2 = \xi_2\} \cup \\ \{(x_1, x_2, x_3) | \, x_1 = \xi_1 + t\alpha_1, \, x_2 \leq \xi_2\} \\ \cup \{(x_1, x_2, x_3) | \, x_1 = \xi_1 + t\beta_1, \, x_2 \geq \xi_2\} \, . \end{aligned}$$

$2d$ and $3d$ Kelvin-Helmholtz problem - a comparison

- In the problem under consideration the vorticity is concentrated on an orientable curve, in $2d$, or a surface in $3d$.

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- In the second example the function $x_2 \mapsto u_1(x_2)$ does not seem to require more than C^1 regularity in order to maintain this regularity. For the two-dimensional Kelvin-Helmholtz (Birkhoff-Rott) such property is not possible, while it might be possible in the three-dimensional case.

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Let $\Gamma(t)$ be the vortex-sheet, the interface where the vorticity is concentrated. The velocity outside interface $\Gamma(t)$ is given by:

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$$\partial_t y - v_2 = -(v_1 \partial_x y)\,,$$
$$\partial_t \tilde{\omega} + \partial_x (v_1 \Omega_0) = -\epsilon \partial_x (v_1 \tilde{\omega})\,,$$
$$v_1(x,t) = -\frac{1}{2\pi} P.V. \int \frac{y(x,t) - y(x',t)}{(x-x')^2 + \epsilon^2 (y(x,t) - y(x',t))^2} (\Omega_0 + \epsilon \tilde{\omega}) dx'\,,$$
$$v_2(x,t) = \frac{1}{2\pi} P.V. \int \frac{x - x'}{(x-x')^2 + \epsilon^2 (y(x,t) - y(x',t))^2} (\Omega_0 + \epsilon \tilde{\omega}) dx'\,.$$

This system describes perturbations in $\mathbb{R}^2$ about the stationary solution

$$y(x,0) = 0\,, u_- = \frac{\Omega_0}{2}\,, u_+ = -\frac{\Omega_0}{2}\,.$$

Local ellipticity continue ...

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&Hf(x) = \frac{1}{\pi} \int \frac{1}{x - x'} f(x') dx' = F^{-1}(-i \mathrm{sgn}(\xi) \hat{f}(\xi)) \\
&|D| f(x) = \frac{1}{\pi} P.V. \int \frac{f(x) - f(x')}{(x-x')^2} = \partial_x (Hf(x)) = F^{-1}(|\xi|) \hat{f}(\xi)) \, .
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$$\partial_{tt}(y_x) + \Omega_0^2 \partial_{xx}(y_x) = \epsilon(\partial_t(F(y_x, \omega)_x) + |D|(\epsilon G(y_x, \omega)_x)\,,$$
$$\partial_{tt}(\omega) + \Omega_0^2 \partial_{xx}(\omega) = \epsilon(|D|(F(y_x, \omega)_x) + \partial_t(\epsilon G(y_x, \omega)_x)\,.$$

What is the situation in the three-dimensional Kelvin-Helmholtz?

Repeat the previous analysis for

$$\Gamma(t) = \{x_3 = \epsilon x(x_1, x_2, t)\}$$

,
a *small* perturbation about the stationary flat state
$x_3 = 0, \tilde{\omega}^0(x_1, x_2) = (\tilde{\omega}_1^0, \tilde{\omega}_2^0, 0)$.

Leading part of the perturbed equation $\partial_t \begin{pmatrix} \hat{x_3} \\ \hat{\omega_1} \\ \hat{\omega_2} \\ \hat{\omega_3} \end{pmatrix} = \mathcal{A} \begin{pmatrix} \hat{x_3} \\ \hat{\omega_1} \\ \hat{\omega_2} \\ \hat{\omega_3} \end{pmatrix}$

with $k = |k|(\cos\theta, \sin\theta)$ and

$$\mathcal{A} = \begin{pmatrix} 0 & \frac{i}{2}\sin\theta & -\frac{i}{2}\cos\theta & 0 \\ -\frac{i}{2}|k|^2|\omega^0|^2\sin\theta & 0 & 0 & \frac{1}{2}(k\cdot\omega^0)\sin\theta \\ \frac{i}{2}|k|^2|\omega^0|^2\cos\theta & 0 & 0 & -\frac{1}{2}(k\cdot\omega^0)\cos\theta \\ 0 & -\frac{1}{2}(k\cdot\omega^0)\sin\theta & \frac{1}{2}(k\cdot\omega^0)\cos\theta & 0 \end{pmatrix}$$

The eigenvalues of the matrix $\mathcal{A}$ are

$$\{0, 0, -\frac{1}{2}|k \wedge \tilde{\omega}^0|, \frac{1}{2}|k \wedge \tilde{\omega}^0|\}\,.$$

Local analysis of Kelvin-Helmholtz continue ...

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Therefore, the first order pseudo-differential operator

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That is the three-dimensional Kelvin-Helmholtz (Birkhoff-Rott) problem is more stable than the two-dimensional one!!!

Numerical investigation of blow up - Hyperviscosity

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Does not develop singularity in finite time. How about the hyper-viscous Hamilton-Jacobi:

$$\frac{\partial u}{\partial t} + \nu \Delta^{2m} u = |\nabla u|^2 ?$$

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This equation blows up in finite time Constantin [Commun. Math. Phys. 1986].

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This result of Chae does not rule out the singularity observed, computationally, by Hou and Luo.

THANK YOU FOR YOUR ATTENTION!