Combinatorics of exclusion processes with open boundaries

Sylvie Corteel (CNRS Paris 7)

Clay Institute, Oxford, May 2015

Koornwinder moments and the two species ASEP

Sylvie Corteel (CNRS Paris 7)

Lauren Williams (Berkeley) Arxiv 1505...

Triangular staircase tableaux

Sylvie Corteel, Olya Mandelshtam (Berkeley) and Lauren Williams (Berkeley)

Fondation Sciences Mathématiques de Paris, Bourse Chateaubriand and France Berkeley Fund

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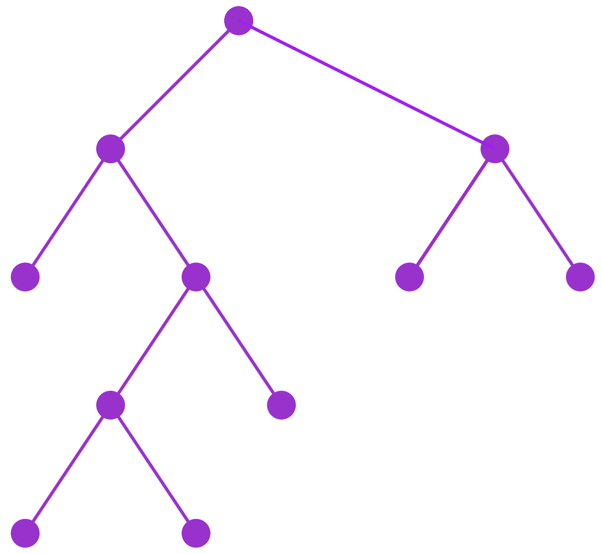
Binary trees, Paths and tableaux



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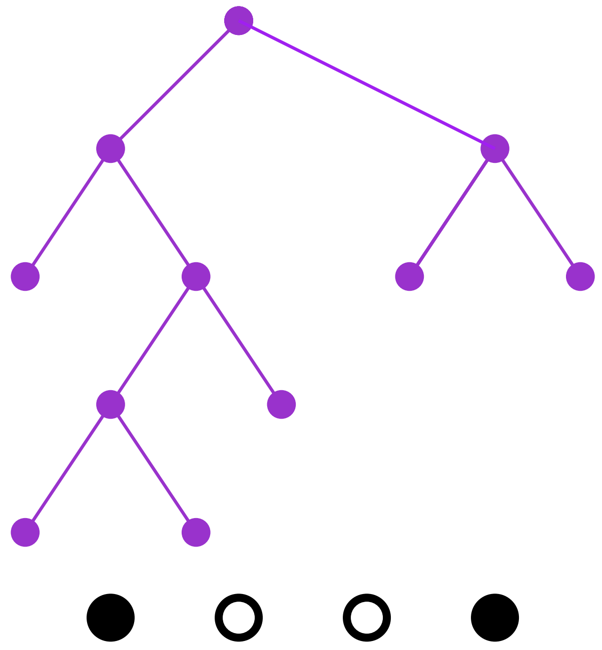

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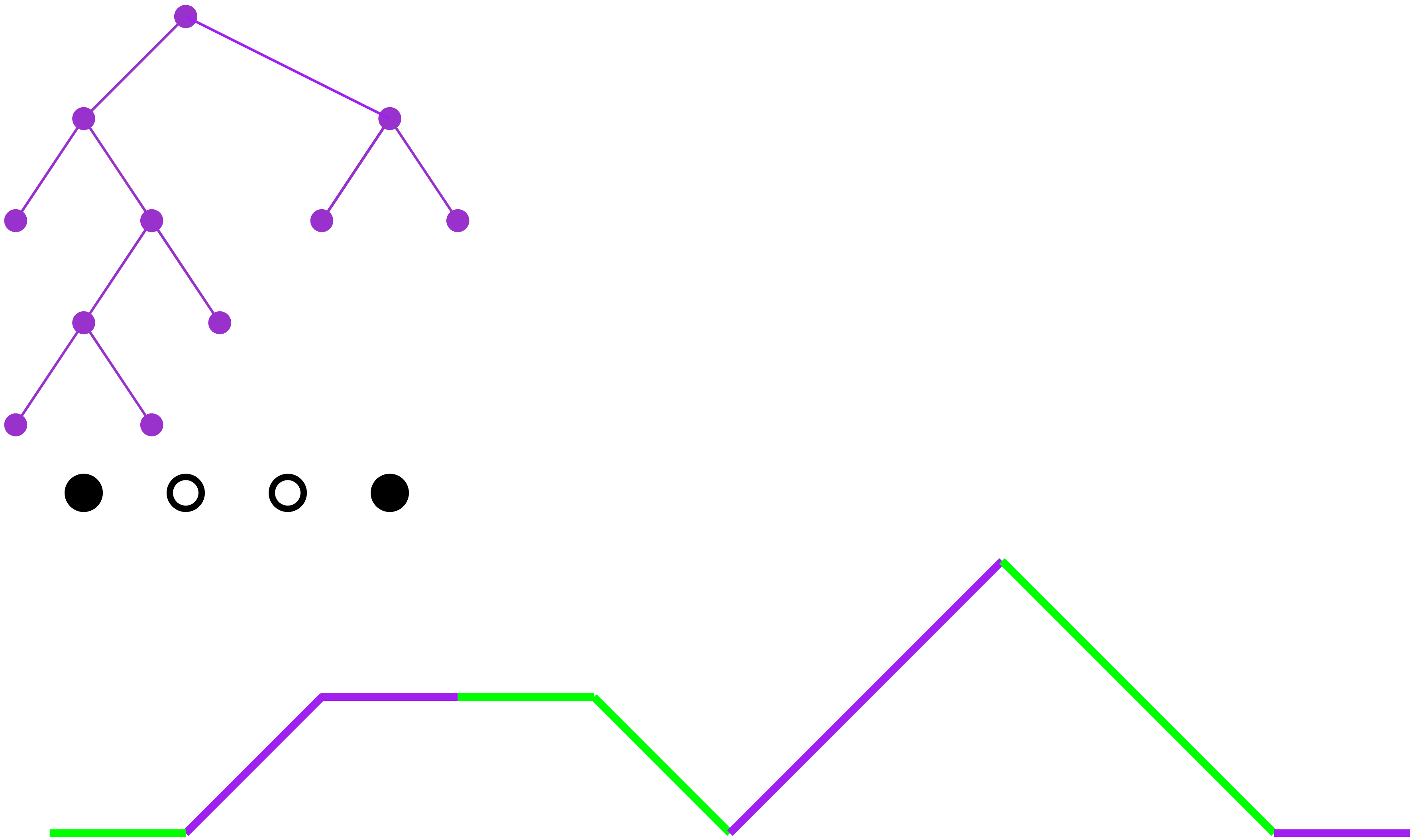

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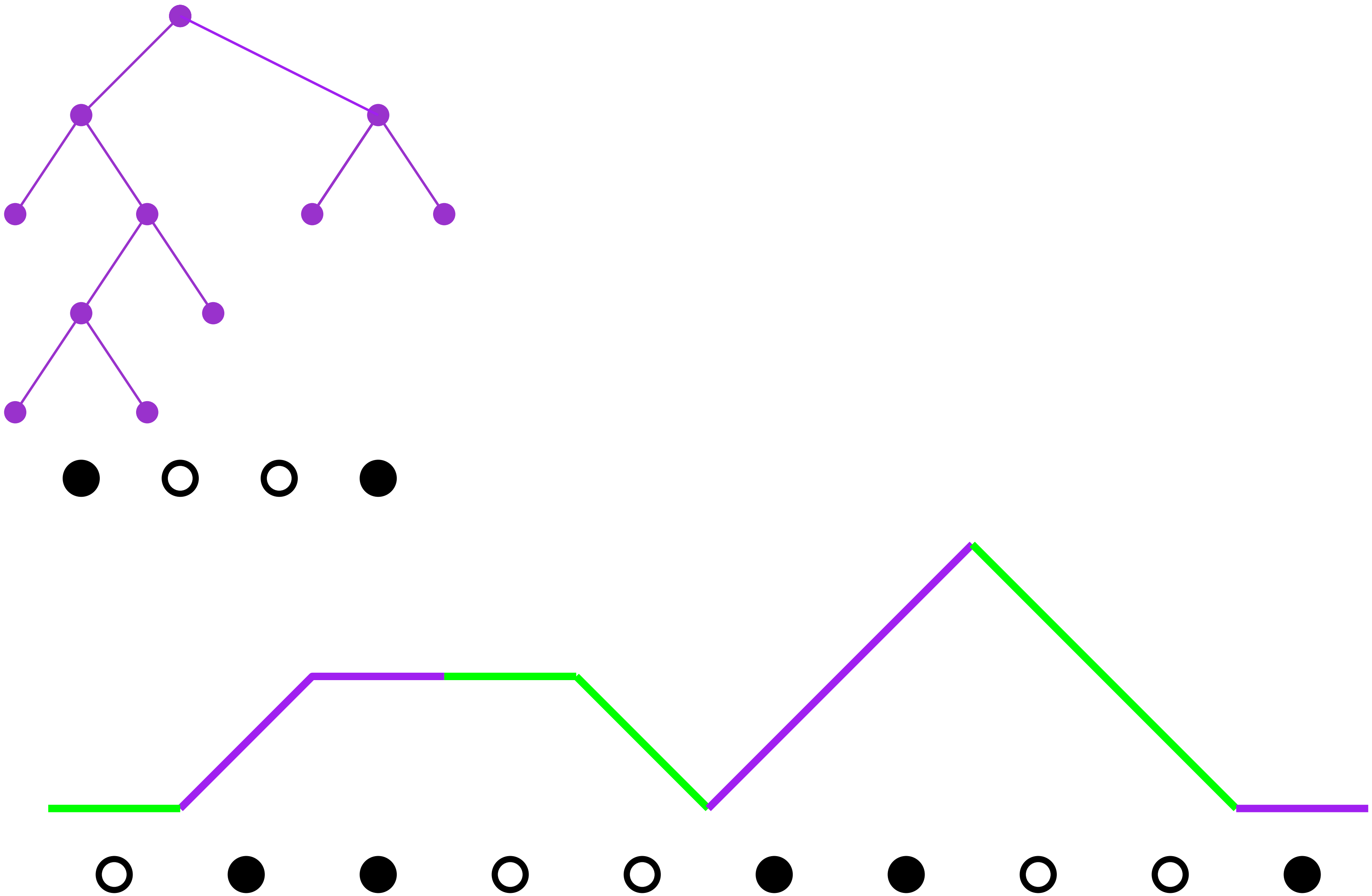

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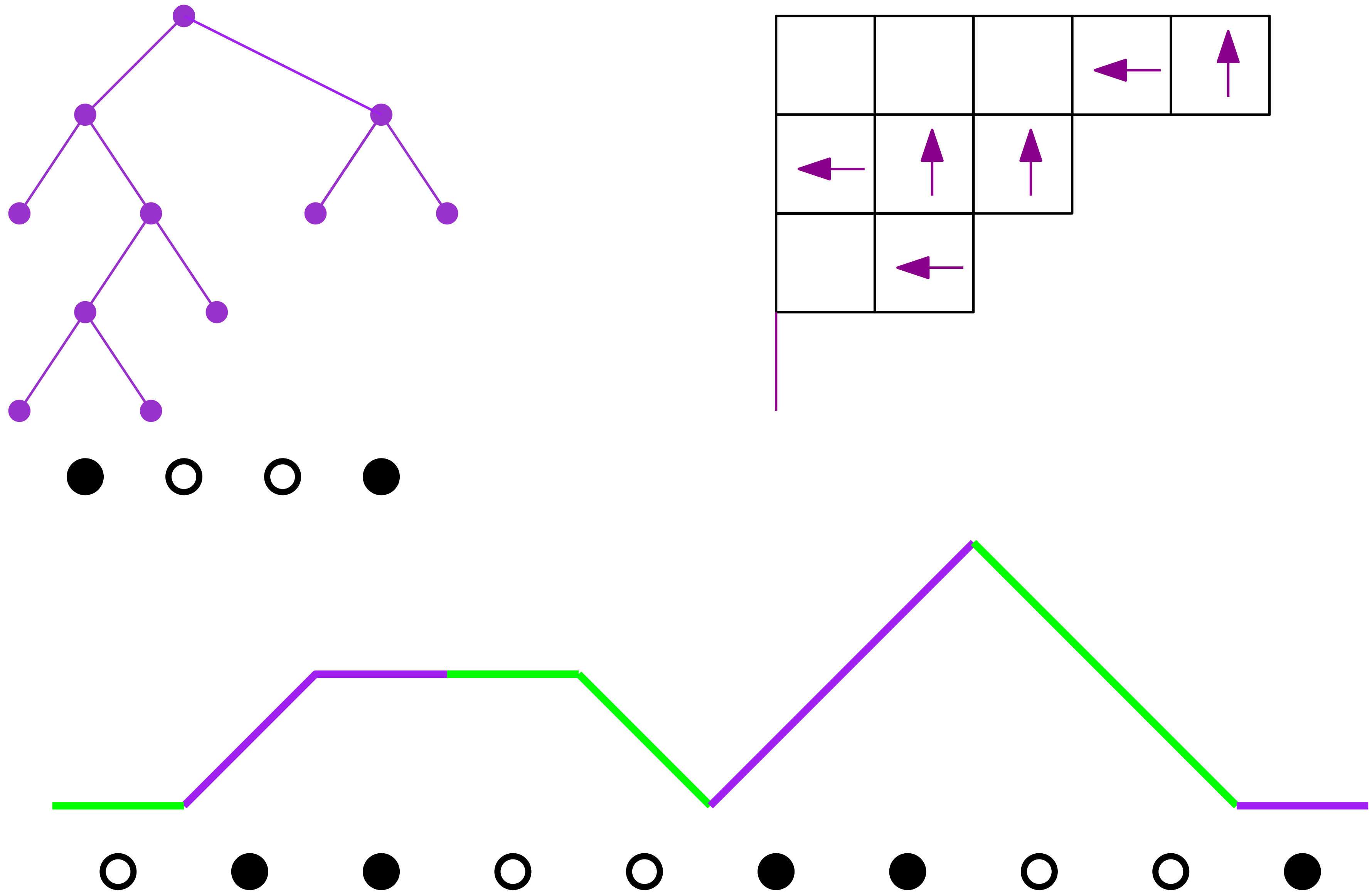

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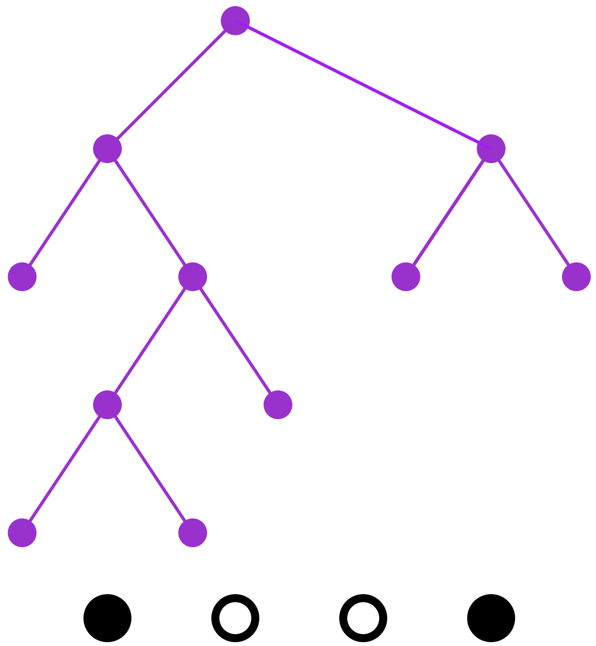
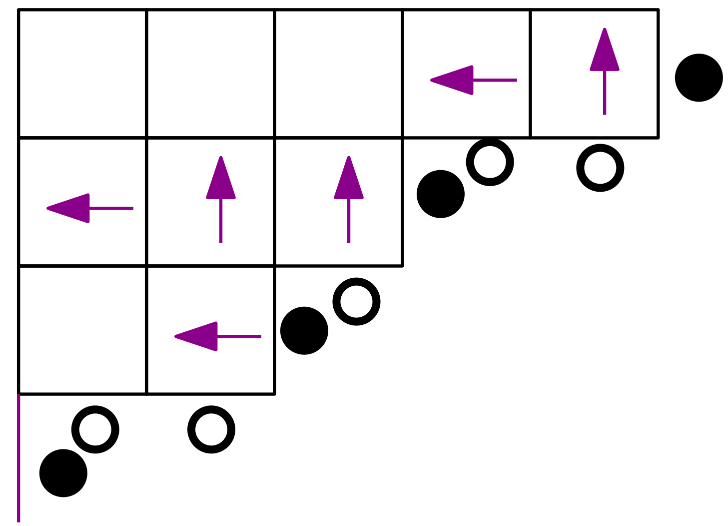
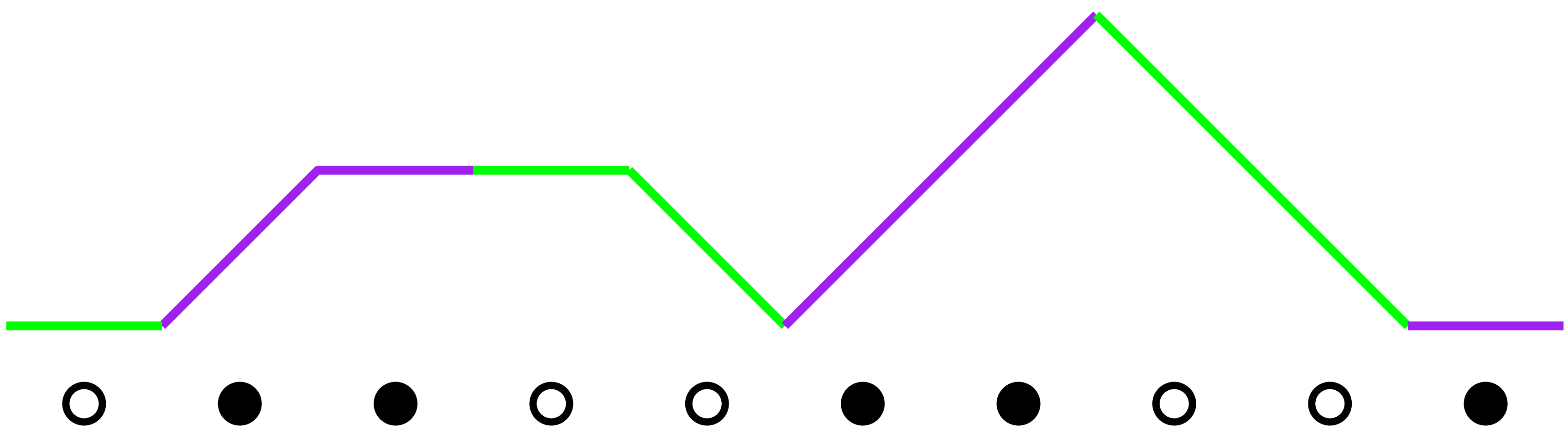






Binary trees, Paths and tableaux

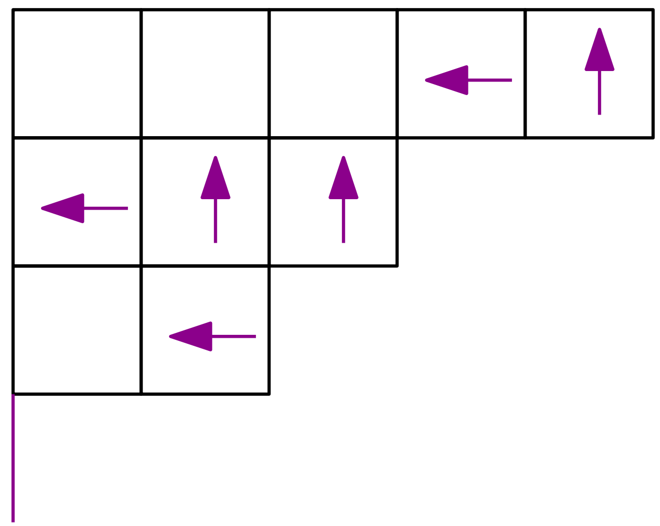
$\tau \in \{\circ, \bullet\}^N$

$B(\tau)$ number of trees of canopy τ

$M(\tau)$ number of paths of shape τ

$C(\tau)$ number of tableaux of shape τ

Binary trees, Paths and tableaux



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$$B(\tau) = M(\tau) = C(\tau)$$

$\sum_\tau C(\tau) = C_{n+1}$ Catalan numbers

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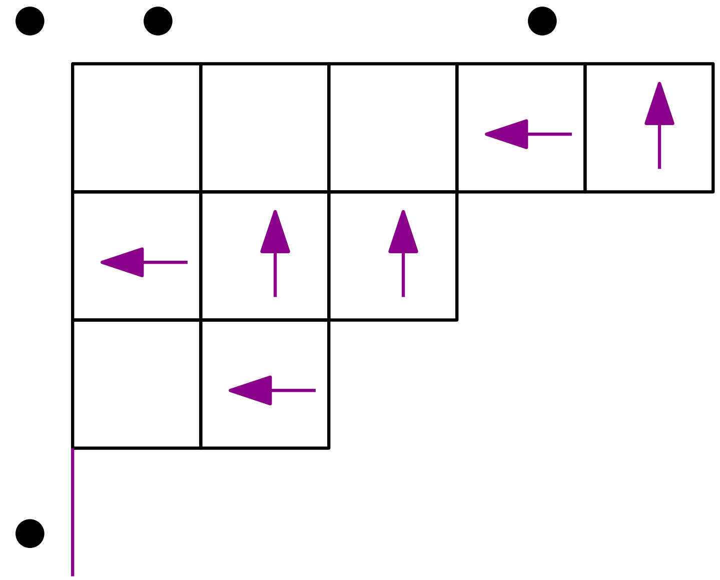

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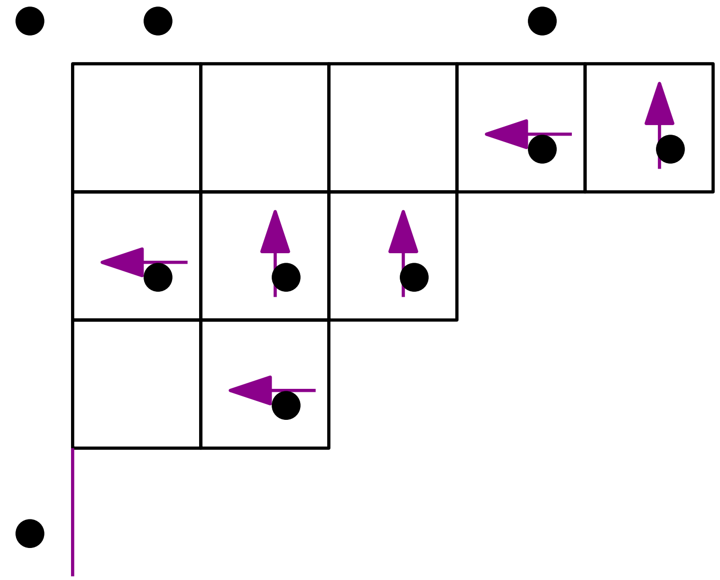

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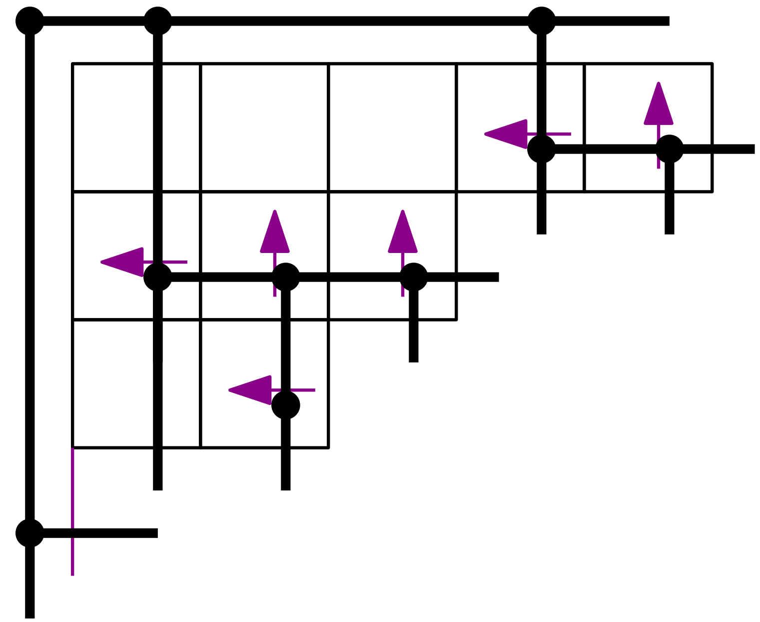

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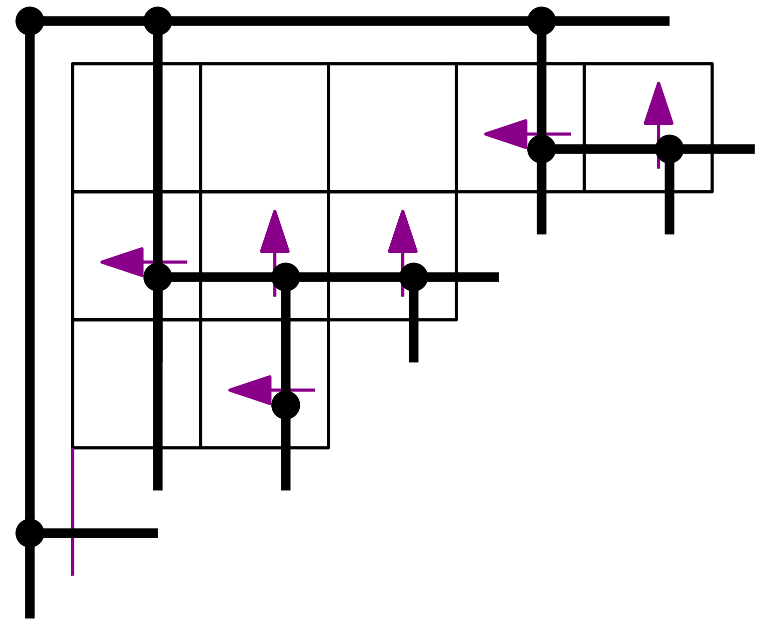

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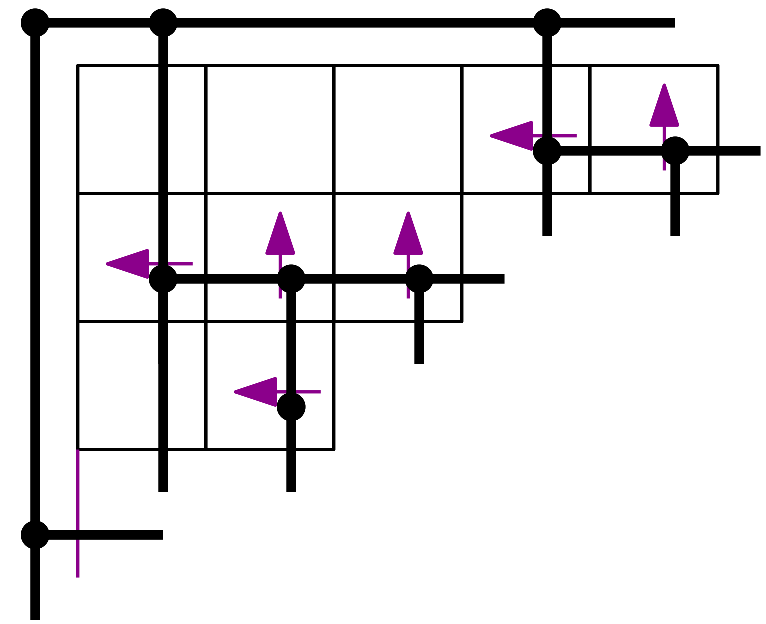

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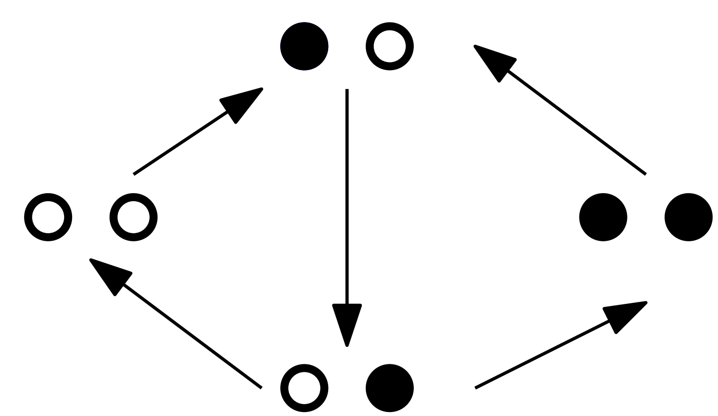

$\sum_\tau C(\tau) = C_{n+1}$ Catalan numbers

Matrix Ansatz [Derrida et al 93]

Matrices D and E, and vectors $\langle W|$ and $|V\rangle$

- $\langle W|E = \langle W|$
- $D|V\rangle = |V\rangle$
- $DE = D + E$

$$Z_N = \langle W|(D+E)^N|V\rangle.$$

Steady state $\tau \in \{\circ, \bullet\} = \{0,1\}^N$

$$P(\tau) = \frac{\langle W| \prod_{i=1}^N [\tau_i D + (1-\tau_i)E] |V\rangle}{Z_N}.$$

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Solution: $\langle W| = (1, 0, \ldots)$, $|V\rangle = (1, 0, \ldots)^T$

$$D = \begin{pmatrix} 1 & 1 & 0 & 0 & \ldots \\ 0 & 1 & 1 & 0 & \ldots \\ 0 & 0 & 1 & 1 & \ldots \\ 0 & 0 & 0 & 1 & \ldots \\ \vdots & & & & \vdots \end{pmatrix} \qquad E = \begin{pmatrix} 1 & 0 & 0 & 0 & \ldots \\ 1 & 1 & 0 & 0 & \ldots \\ 0 & 1 & 1 & 0 & \ldots \\ 0 & 0 & 1 & 1 & \ldots \\ \vdots & & & & \vdots \end{pmatrix}$$

Motzkin paths [Zeilberger, Duchi and Schaeffer, Brak and Essam]

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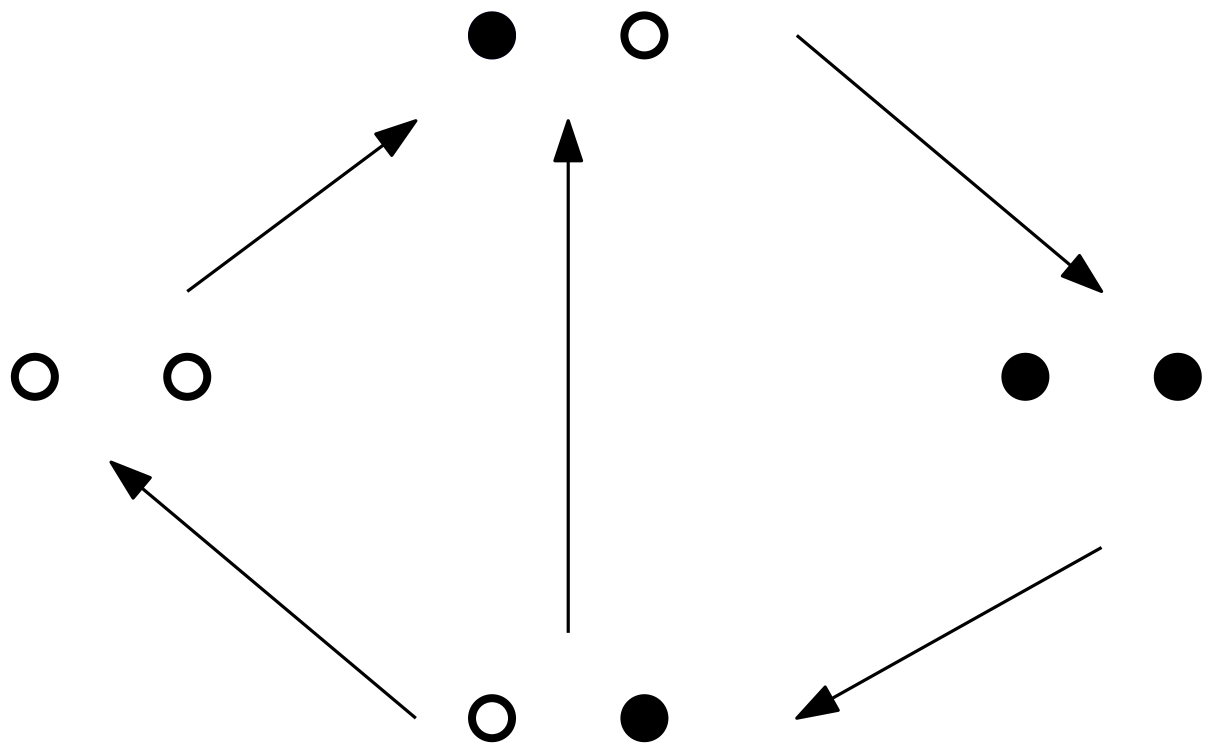
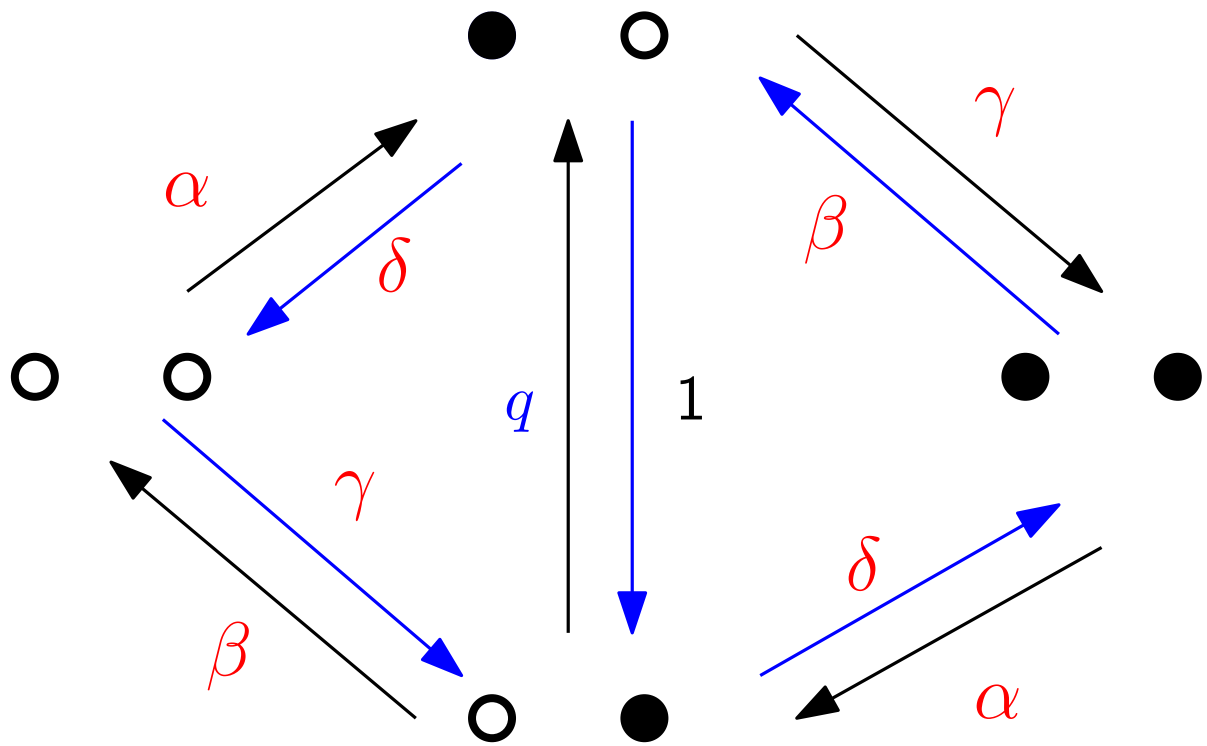
Lukasiewicz paths, Catalan tableaux

Asymmetric exclusion process with 5 parameters



Asymmetric exclusion process with 5 parameters



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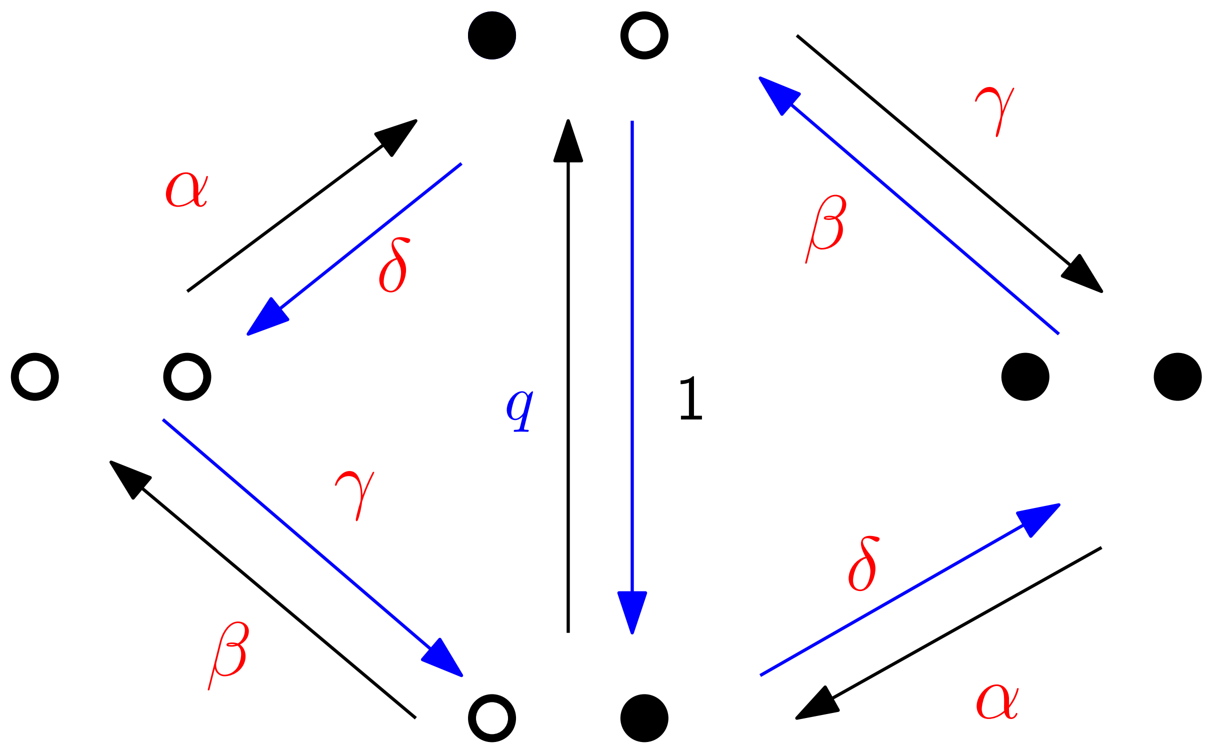
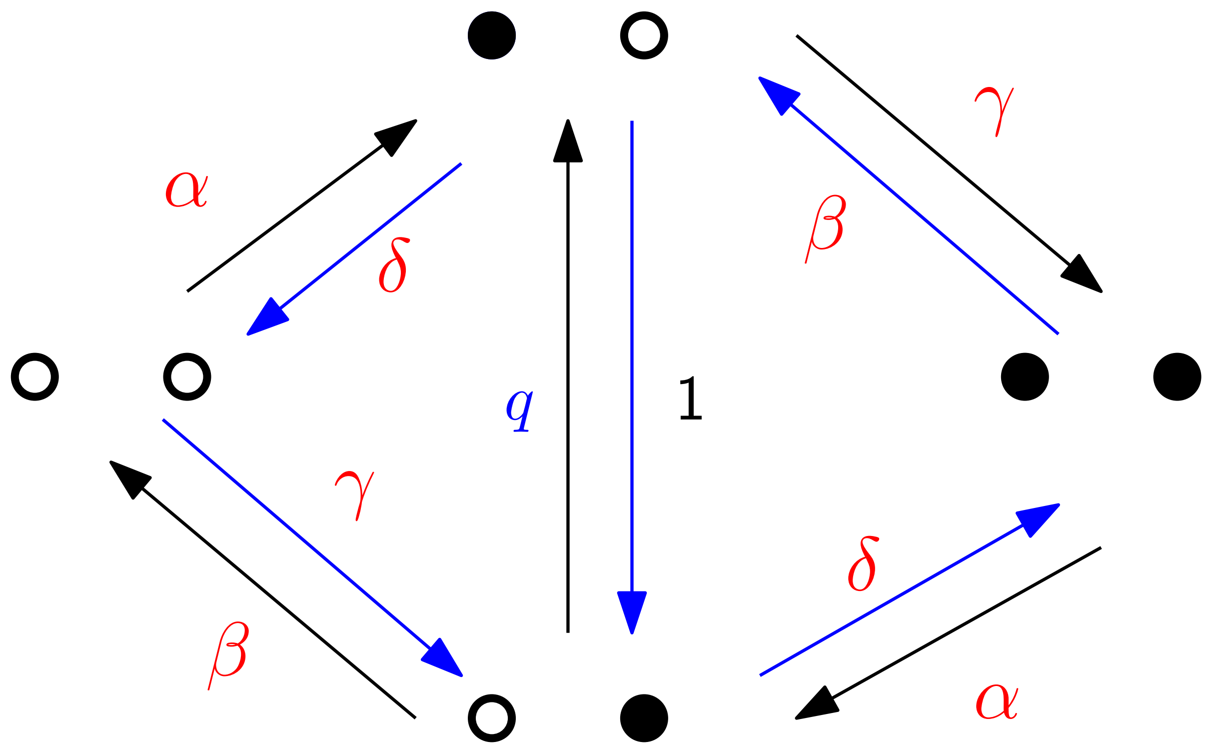


Matrix Ansatz

- $\langle W|(\alpha E - \gamma D) = \langle W|$
- $(\beta D - \delta E)|V\rangle = |V\rangle$
- $DE = qED + D + E$

Asymmetric exclusion process with 5 parameters



Matrix Ansatz

- $\langle W|(\alpha E - \gamma D) = \langle W|$
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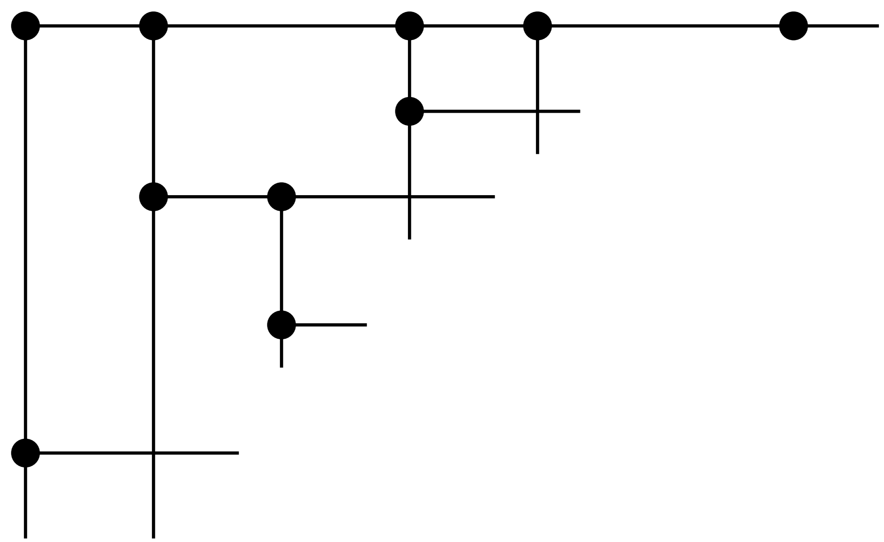
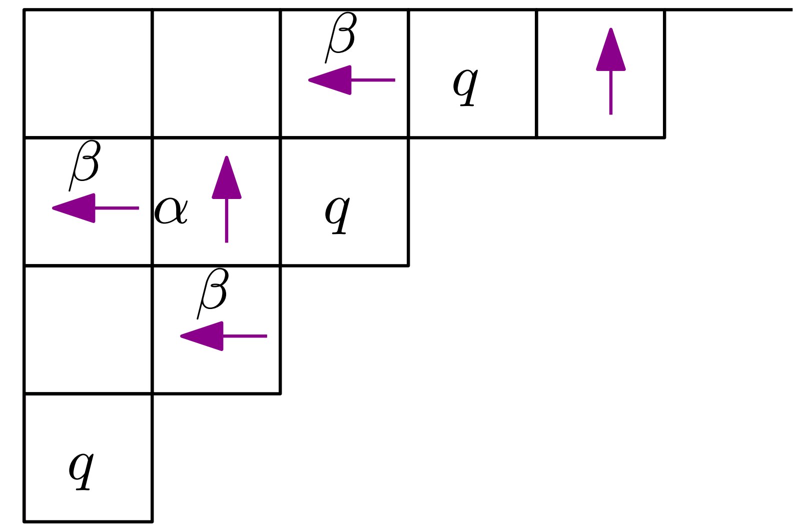
- $DE = qED + D + E$

$\gamma = \delta = 0$

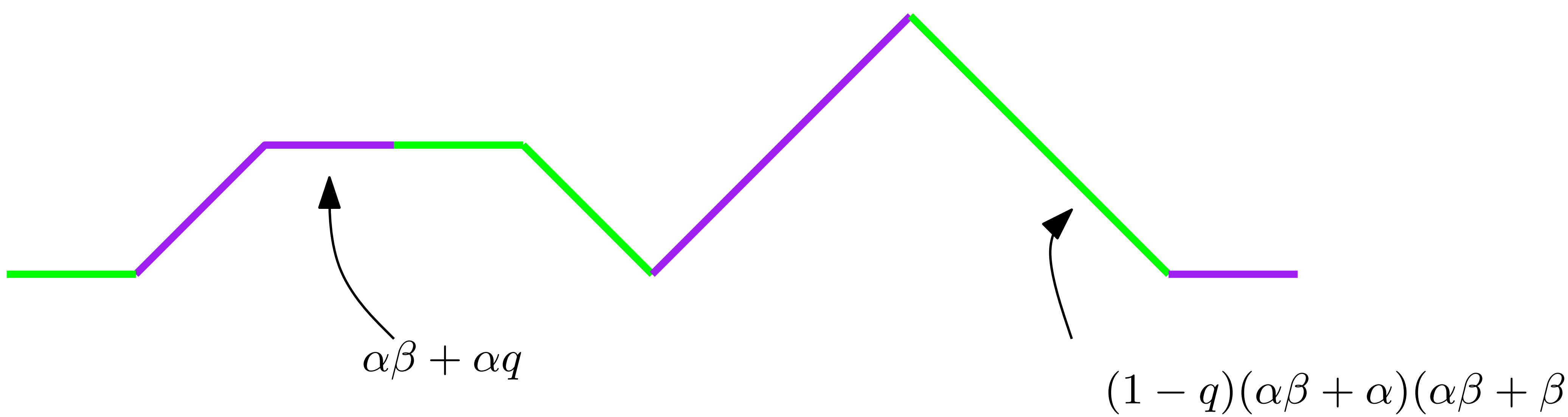
- Trees $\Rightarrow$ tree like tableaux
- Paths $\Rightarrow$ moments of AlSalam-Chihara Polynomials
- Tableaux $\Rightarrow$ Permutation tableaux, Alternative tableaux

Tree like tableaux, alternative tableaux, labelled paths





[Aval, Boussicault, C. Josuat-Vergès, Nadeau, Viennot, Williams. . .]



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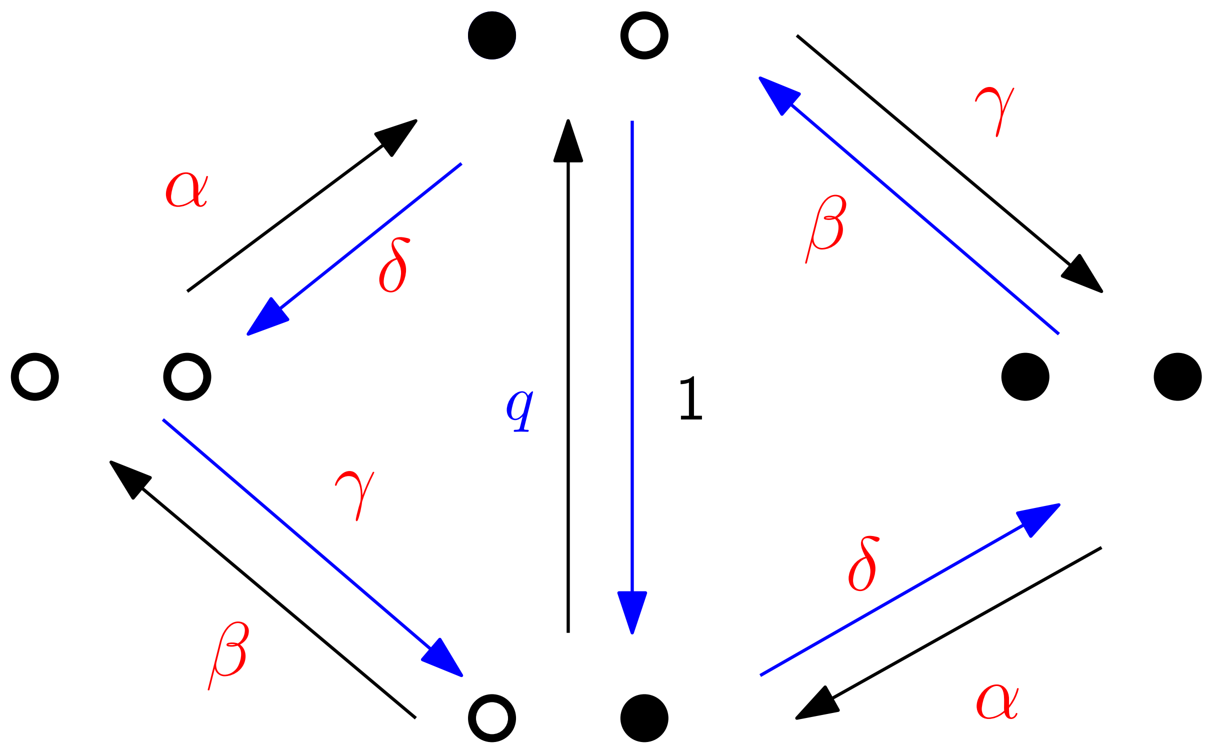

General model

- Moments of Askey Wilson polynomials [Uchiyama, Sasamoto, Wadati 04]
- Staircase tableaux [C., Williams 10]

Askey Wilson polynomials

$$P_{n+1}(x) = (x - b_n)P_n(x) - \lambda_n P_{n-1}(x)$$

$$b_n = 1/2(a + 1/a - A_n - C_n) \qquad \lambda_n = A_{n-1}C_n/4$$

$$A_n = \frac{(1-abq^n)(1-acq^n)(1-adq^n)(1-abcdq^{n-1})}{a(1-abcdq^{2n})(1-abcdq^{2n-1})}$$

symmetric in a, b, c, d

$$C_n = \frac{(1-abq^{n-1})(1-bcq^{n-1})(1-bdq^{n-1})(1-q^n)}{a(1-abcdq^{2n-2})(1-abcdq^{2n-1})}$$

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orthogonal

$$\oint_C \frac{dz}{4\pi iz} w\left(\frac{z+z^{-1}}{2}\right) P_m\left(\frac{z+z^{-1}}{2}\right) P_n\left(\frac{z+z^{-1}}{2}\right) = h_n \delta_{mn},$$

$$w(x) = \frac{(z^2,z^{-2};q)_\infty}{(az,a/z,bz,b/z,cz,c/z,dz,d/z;q)_\infty},\ x = (z + z^{-1})/2$$

$$h_n = \frac{(1-q^{n-1}abcd)(q,ab,ac,ad,bc,bd,cd;q)_n}{(1-q^{2n-1}abcd)(abcd;q)_n} \qquad (a;q)_n = \prod_{i=0}^{n-1}(1 - aq^i)$$

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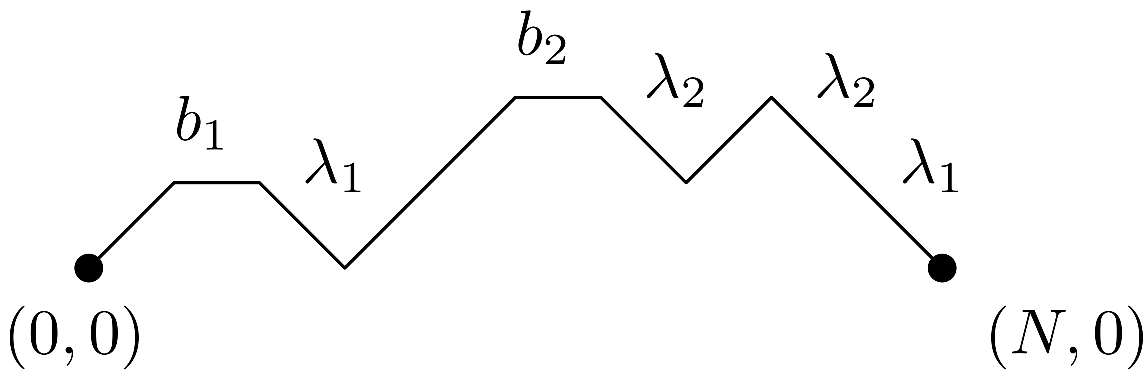
Moments

$$\mu_N^{AW} = \oint_C \frac{dz}{4\pi iz} w\left(\frac{z+z^{-1}}{2}\right)\left(\frac{z+z^{-1}}{2}\right)^N$$

Combinatorics of moments

[Flajolet, Viennot 80s]

$$P_{n+1}(x) = (x - b_n)P_n(x) - \lambda_n P_{n-1}(x)$$



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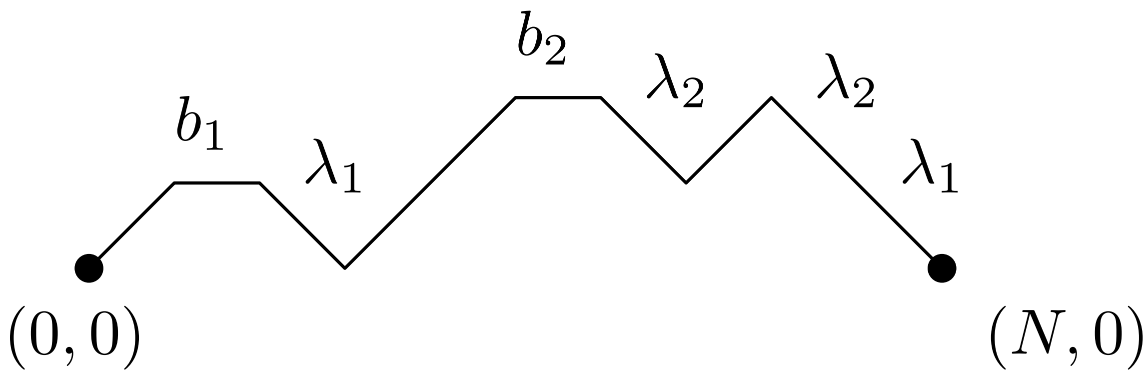

$W(p) = b_1 b_2 \lambda_1^2 \lambda_2^2$

$$\mu_N = \sum_p W(p)$$

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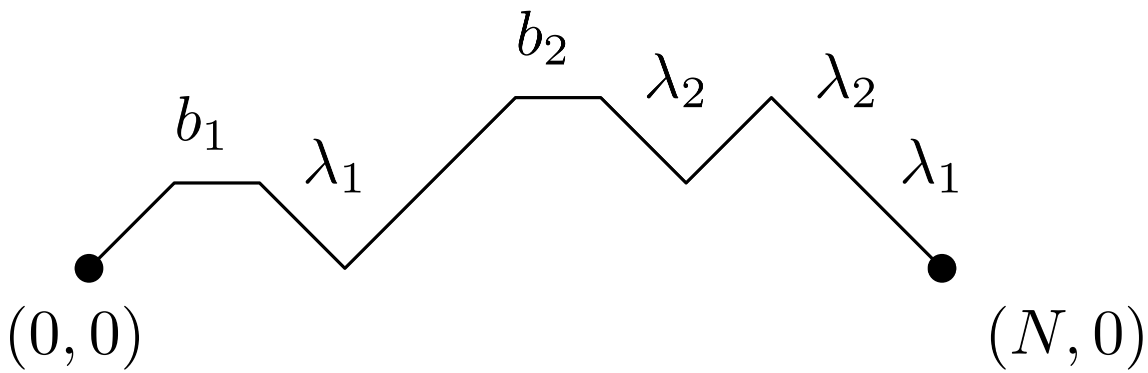

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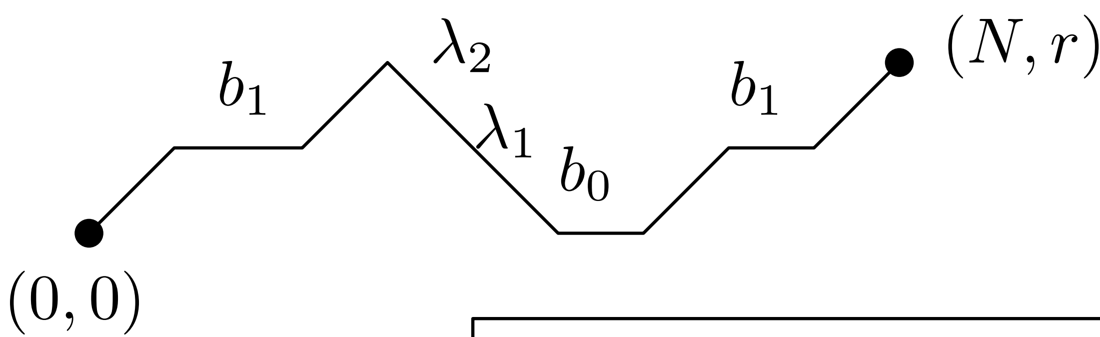

$$\mu_{N,r} = \oint_C \frac{dz}{4\pi i z} w\left(\frac{z+z^{-1}}{2}\right) P_r\left(\frac{z+z^{-1}}{2}\right)\left(\frac{z+z^{-1}}{2}\right)^N$$

Solution of the 5 parameter model [USW 04]

$$\mathsf{d} = \begin{pmatrix} d_0^\natural & d_0^\sharp & 0 & \cdots \\ d_0^\flat & d_1^\natural & d_1^\sharp & \\ 0 & d_1^\flat & d_2^\natural & \ddots \\ \vdots & & \ddots & \ddots \end{pmatrix}$$

$$d_n^\natural + e_n^\natural = b_n$$

$$d_n^\flat = -\frac{q^n bd}{(1-q^n ac)(1-q^n bd)}\lambda_n \qquad e_n^\flat = \frac{1}{(1-q^n ac)(1-q^n bd)}\lambda_n \qquad d_n^\sharp = 1 \;\; e_n^\sharp = -q^n ac$$

$$\langle W| = (1, 0, \ldots),\ |V\rangle = (1, 0, \ldots)^T$$

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$$\langle W| = (1,0,\ldots),\; |V\rangle = (1,0,\ldots)^T$$

$$\boxed{\mu_N^{\mathrm{AW}} = \langle W|(\mathsf{d}+\mathsf{e})^N|V\rangle}$$

$$a = \frac{1-q-\alpha+\gamma+\sqrt{(1-q-\alpha+\gamma)^2+4\alpha\gamma}}{2\alpha} \;\; b = \frac{1-q-\beta+\delta+\sqrt{(1-q-\beta+\delta)^2+4\beta\delta}}{2\beta}$$

$$D = \frac{1+\mathsf{d}}{1-q},\; E = \frac{1+\mathsf{e}}{1-q}$$

$$\boxed{Z_N = \langle W|(D+E)^N|V\rangle}$$

Koorwinder polynomials

Multivariate version of the AW polynomials $P_\lambda(z_1,\ldots,z_m;a,b,c,d|q,t)$

at $q=t$

$$P_\lambda(\mathrm{z};a,b,c,d|q,q) = \mathrm{const}\cdot\frac{\det(p_{m-j+\lambda_j}(z_i;a,b,c,d|q))_{i,j=1}^m}{\det(p_{m-j}(z_i;a,b,c,d|q))_{i,j=1}^m}$$

Density

$$\prod_{1\le i<j\le m}(1-z_iz_j)(1-z_i/z_j)(1-z_j/z_i)(1-1/z_iz_j)\prod_{1\le i\le m} w\left(\frac{z_i+z_i^{-1}}{2}\right)$$

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AW-polynomials

at $q = t$

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Density

$$\prod_{1\le i<j\le m}(1 - z_i z_j)(1 - z_i/z_j)(1 - z_j/z_i)(1 - 1/z_i z_j) \prod_{1\le i\le m} w\left(\frac{z_i + z_i^{-1}}{2}\right)$$

AW-density

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AW-density

Possible definition of moments

$$M_\lambda = I_K(s_\lambda(x_1,\dots,x_m);a,b,c,d;q,q).$$

Koorwinder polynomials

$$\lambda_1 \geq \ldots \geq \lambda_m \geq 0$$

Multivariate version of the AW polynomials $P_\lambda(z_1, \ldots, z_m; a, b, c, d|q, t)$

AW-polynomials

at $q = t$

$$P_\lambda(\mathrm{z}; a, b, c, d|q, q) = \mathrm{const} \cdot \frac{\det(p_{m-j+\lambda_j}(z_i;a,b,c,d|q))_{i,j=1}^m}{\det(p_{m-j}(z_i;a,b,c,d|q))_{i,j=1}^m}$$

Density

$$\prod_{1\leq i<j\leq m}(1 - z_i z_j)(1 - z_i/z_j)(1 - z_j/z_i)(1 - 1/z_i z_j) \prod_{1\leq i\leq m} w\left(\frac{z_i + z_i^{-1}}{2}\right)$$

AW-density

Possible definition of moments

$$M_\lambda = I_K(s_\lambda(x_1, \ldots, x_m); a, b, c, d; q, q).$$

Schur functions

Integrate with respect to the Koorwinder density

Koorwinder polynomials

Multivariate version of the AW polynomials $P_\lambda(z_1,\ldots,z_m;a,b,c,d|q,t)$

AW-polynomials

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Density

$$\prod_{1\le i<j\le m}(1-z_iz_j)(1-z_i/z_j)(1-z_j/z_i)(1-1/z_iz_j)\prod_{1\le i\le m} w\left(\frac{z_i+z_i^{-1}}{2}\right)$$

AW-density

Possible definition of moments

$$M_\lambda = I_K(s_\lambda(x_1,\ldots,x_m);a,b,c,d;q,q).$$

Lemma

Rains

$$M_\lambda = \frac{\det(\mu_{\lambda_i+m-i+m-j})_{i,j=1}^m}{\det(\mu_{2m-i-j})_{i,j=1}^m}$$

Koorwinder moments

$$M_\lambda = \frac{\det(\mu_{\lambda_i+m-i+m-j})_{i,j=1}^{m}}{\det(\mu_{2m-i-j})_{i,j=1}^{m}}$$

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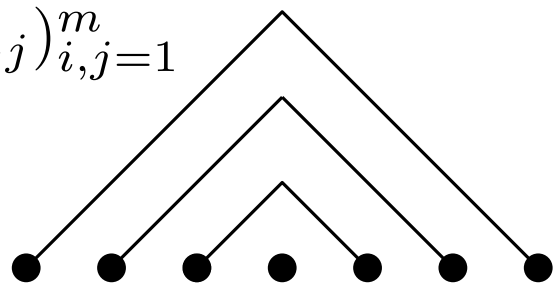
Path interpretation

$\det(\mu_{2m-i-j})_{i,j=1}^m$



Lindström, Gessel, Viennot

$\prod_{i=1}^m \lambda_i^{m-i}$

Koorwinder moments

$$M_\lambda = \frac{\det(\mu_{\lambda_i+m-i+m-j})_{i,j=1}^m}{\det(\mu_{2m-i-j})_{i,j=1}^m}$$

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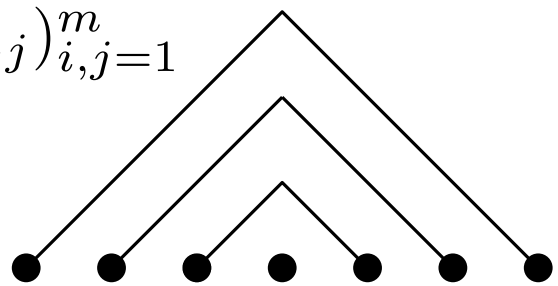
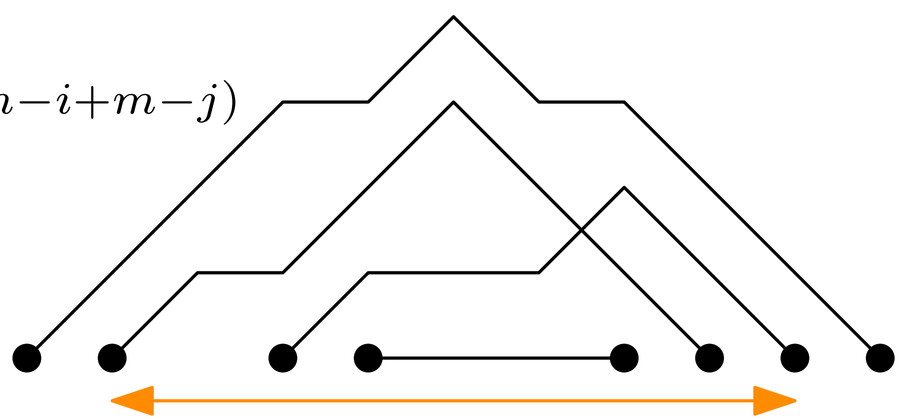


Lindström, Gessel, Viennot

$\prod_{i=1}^m \lambda_i^{m-i}$

$\det(\mu_{\lambda_i+m-i+m-j})$

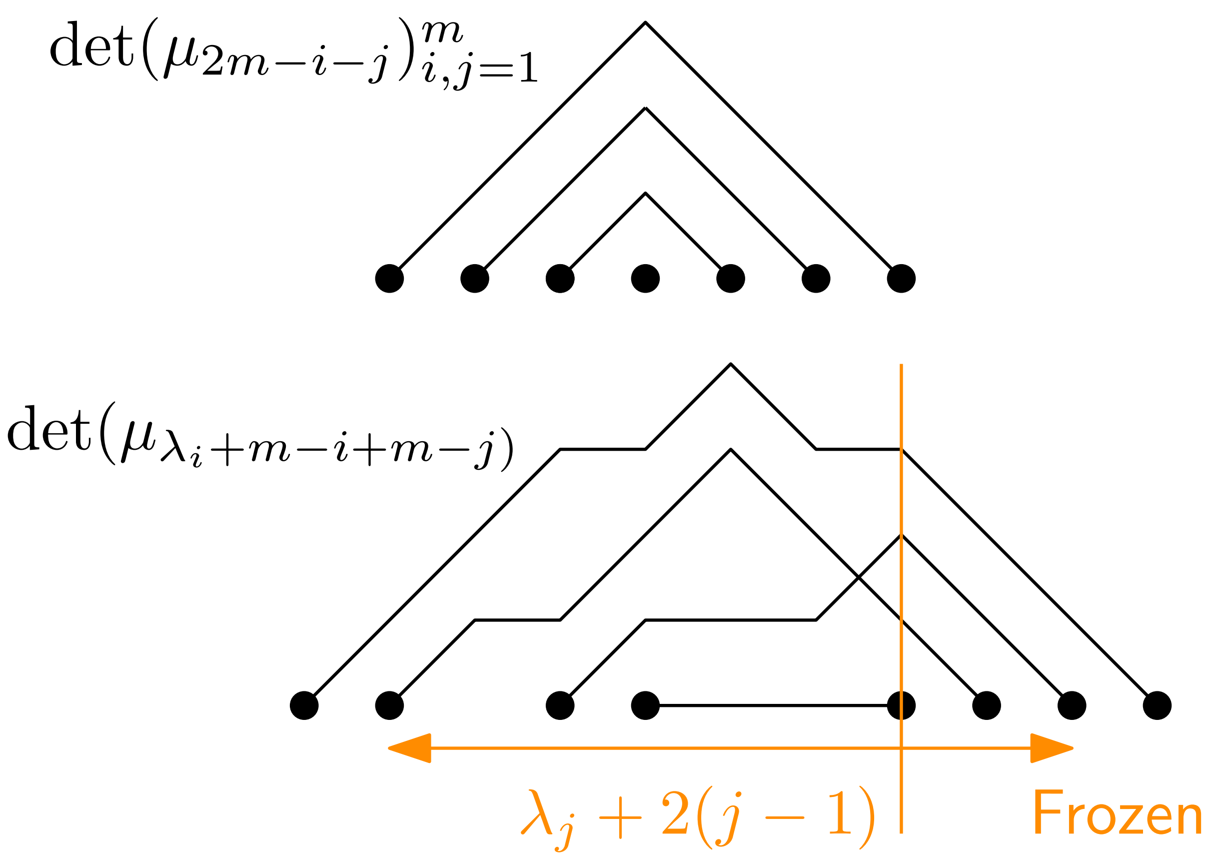


$\lambda_j + 2(j-1)$

Koorwinder moments

$$M_\lambda = \frac{\det(\mu_{\lambda_i+m-i+m-j})_{i,j=1}^{m}}{\det(\mu_{2m-i-j})_{i,j=1}^{m}}$$

Path interpretation



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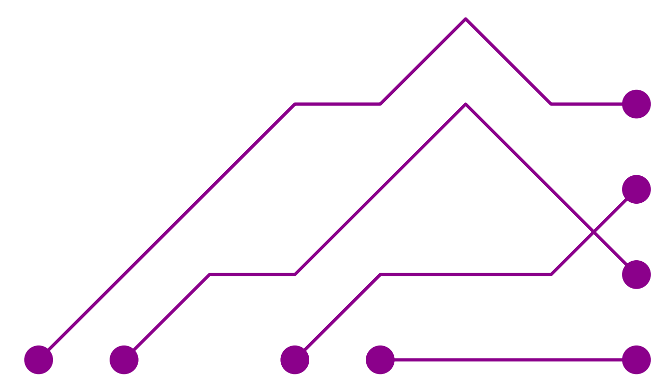

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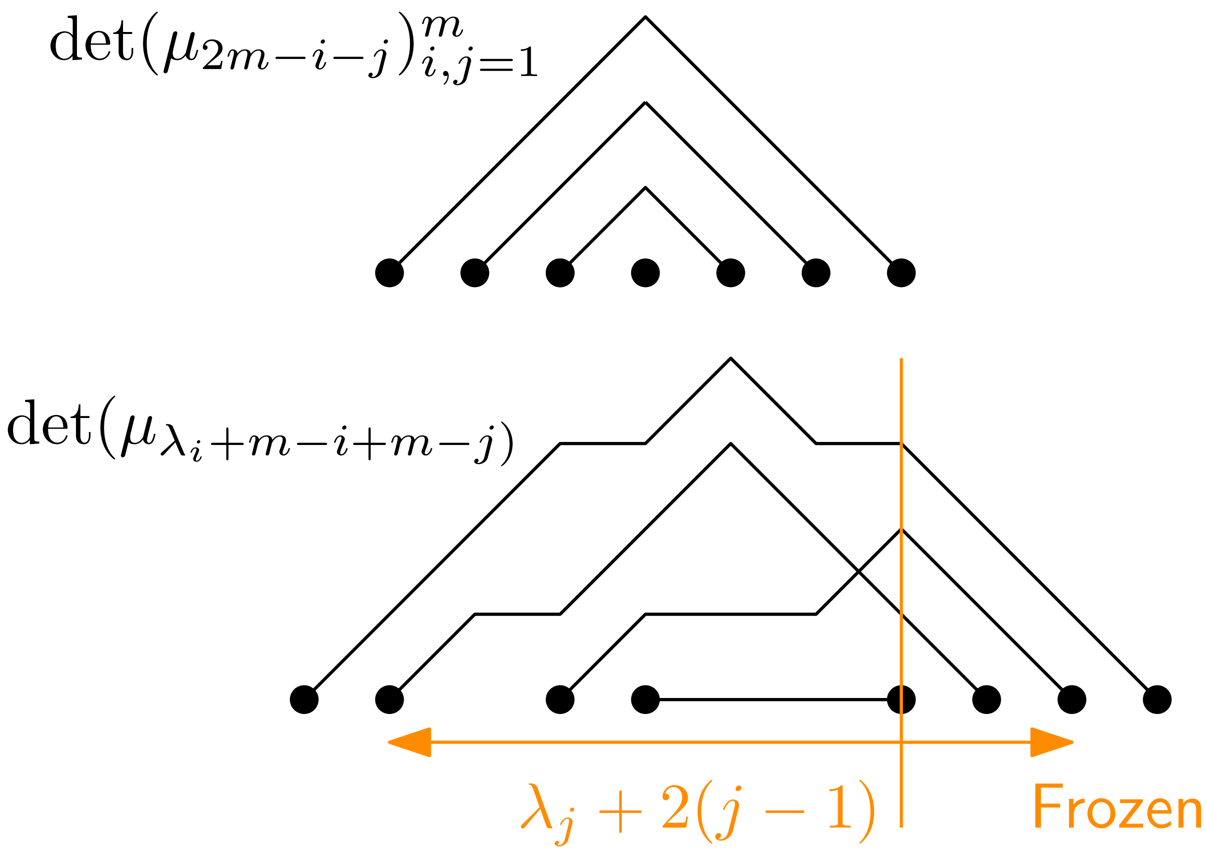
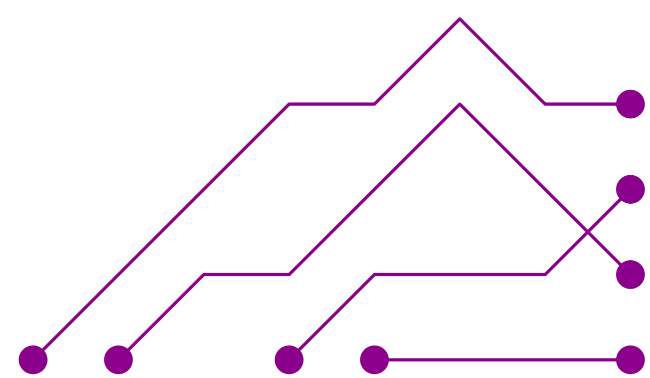


Lindström, Gessel, Viennot

$\prod_{i=1}^m \lambda_i^{m-i}$

$$M_\lambda = \det(\mu_{\lambda_i+n-i+m-j,j})$$



More Koornwinder moments

$$\lambda_1 \geq \ldots \geq \lambda_m \geq 0$$

$$K_\lambda = \frac{\det(Z_{\lambda_i+m-i+m-j})_{i,j=1}^{m}}{\det(Z_{2m-i-j})_{i,j=1}^{m}}$$

$$K_\lambda = \det(K_{(\lambda_i+j-i,0,0,\ldots,0)})_{i,j=1}^{n}$$

Conjecture [C., Rains, Williams 14]
The Koornwinder moment K_λ is a polynomial in $\alpha, \beta, \gamma, \delta, q$ with positive coefficients (up to a normalizing factor).

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The Koornwinder moment K_λ is a polynomial in $\alpha, \beta, \gamma, \delta, q$ with positive coefficients (up to a normalizing factor).

True for $q = 1$

$$Z_n = \frac{\prod_{i=0}^{n-1}(\alpha+\beta+\gamma+\delta+i(\alpha+\gamma)(\beta+\delta))}{\alpha\beta-\gamma\delta}$$

$$K_\lambda = \frac{\text{Nice product}}{(\alpha\beta-\gamma\delta)^{|\lambda|}}$$

More Koornwinder moments

$$\lambda_1 \geq \ldots \geq \lambda_m \geq 0$$

$$K_\lambda = \frac{\det(Z_{\lambda_i+m-i+m-j})_{i,j=1}^m}{\det(Z_{2m-i-j})_{i,j=1}^m}$$

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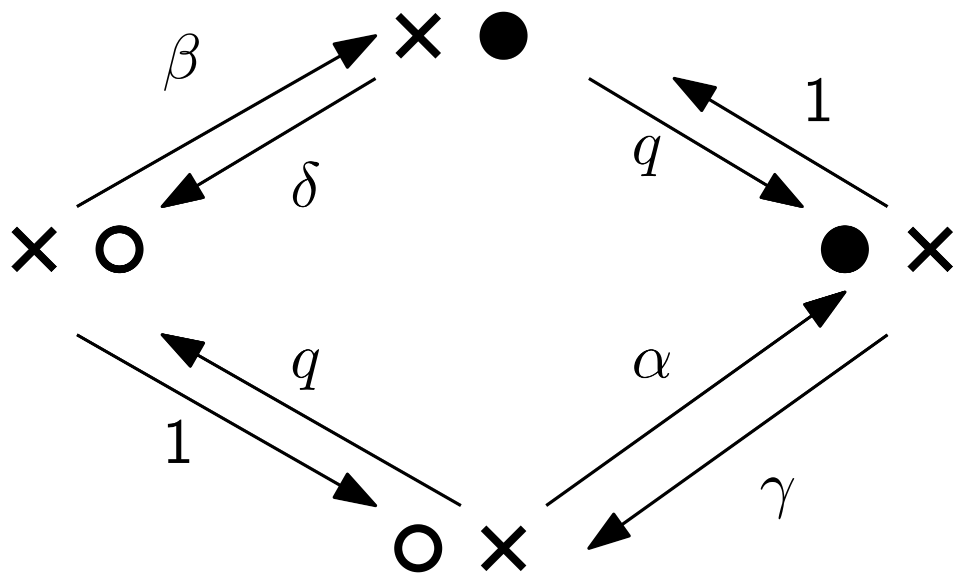
Conjecture [C., Rains, Williams 14]
The Koornwinder moment K_λ is a polynomial in $\alpha, \beta, \gamma, \delta, q$ with positive coefficients (up to a normalizing factor).

True for $\lambda = (N-r, \underbrace{0, \ldots, 0}_{r})$

Theorem [C., Williams 15]
$K_{(N-r,0,\ldots,0)}$ Partition function of the two species ASEP

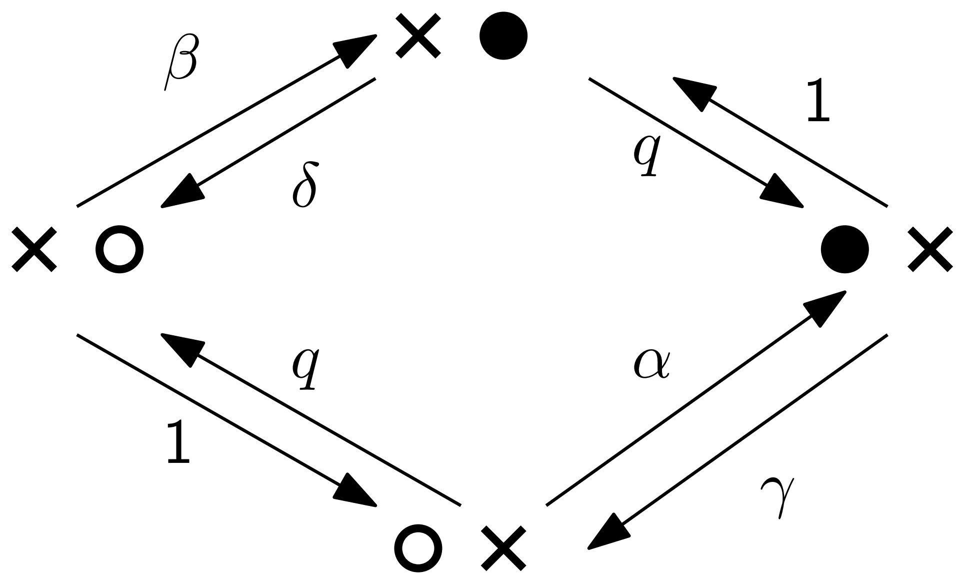
Two species ASEP



N sites

r particles equal to $\times$

Two species ASEP



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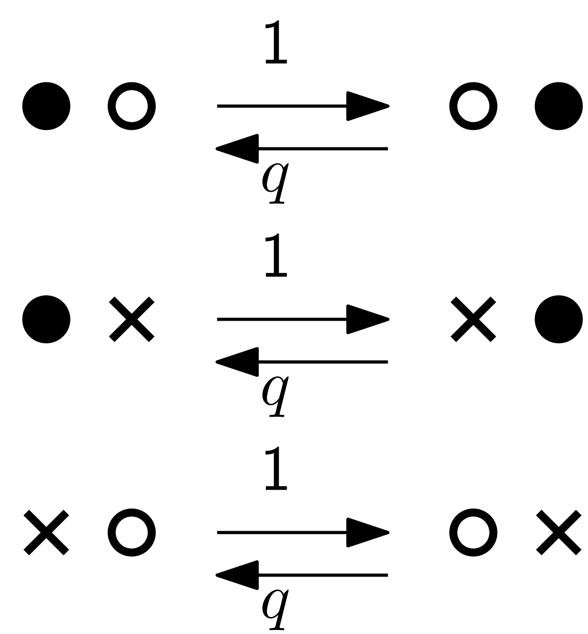
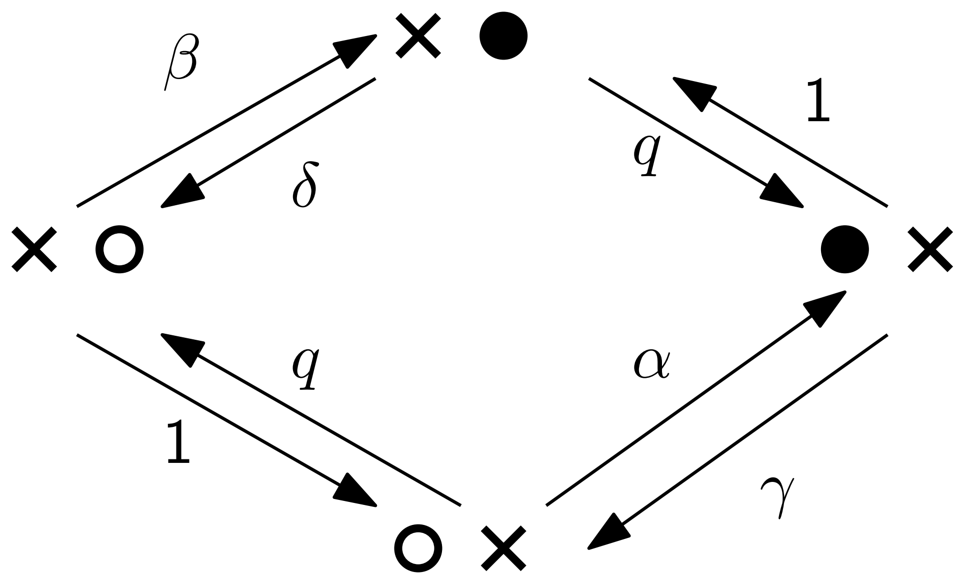


Two species ASEP



N sites

r particles equal to $\times$

Matrix Ansatz [Uchiyama 08]

- $\langle W|(\alpha E - \gamma D) = \langle W|$
- $(\beta D - \delta E)|V\rangle = |V\rangle$
- $DE - qED = D + E$
- $DA = qAD + A$
- $AE = qEA + A.$

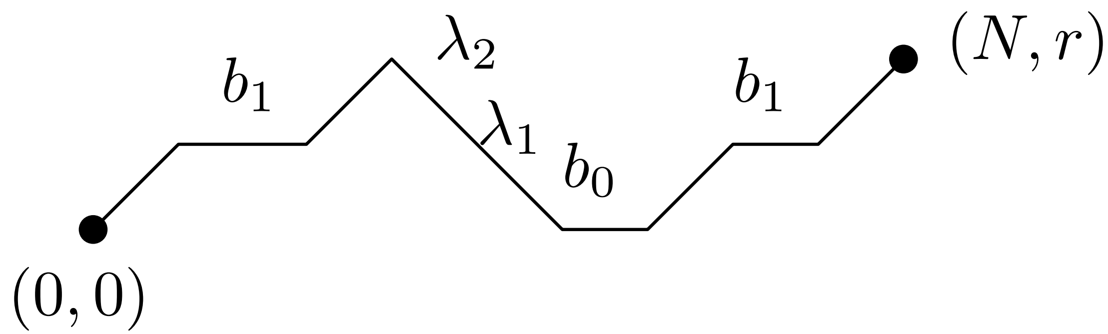
Partition function

$$Z_{N,r} = [y^r]\frac{\langle W|(D+E+yA)^N|V\rangle}{\langle W|A^r|V\rangle}.$$

Two species ASEP

$$Z_{N,r} = [y^r]\frac{\langle W|(D+E+yA)^N|V\rangle}{\langle W|A^r|V\rangle}.$$

$$K_{(N-r.0,\ldots,0)} = \langle W|(D+E)^N|V^r\rangle \qquad |V^r\rangle = (0,\ldots,0,1,0,\ldots)^T$$



$$K_{(N-r,0\ldots,0)} = \mu_{N,r} = \oint_C \frac{dz}{4\pi iz} w\left(\frac{z+z^{-1}}{2}\right) P_r\left(\frac{z+z^{-1}}{2}\right)\left(\frac{z+z^{-1}}{2}\right)^N$$

Theorem. $Z_{N,r} = \text{factor} \times K_{(n-r,0,\ldots,0)}$

Sketch of proof

Lemma. The theorem is true if $\langle W|D^N|V^r\rangle \alpha^r (1-q)^r = [y^r]\frac{\langle W|(D+yA)^N|V\rangle}{\langle W|A^r|V\rangle}$.

Sketch of proof

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Proof. Matrix Ansatz

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Proof. Matrix Ansatz

$D = (1+\mathsf{d})/(1-q)$

Lemma. The theorem is true if $\langle W|\mathsf{d}^N|V^r\rangle = \left[\begin{array}{c} N \\ r \end{array} \right]_q \frac{\langle W|A^r \mathsf{d}^{N-r}|V\rangle}{\langle W|A^r|V\rangle}$.

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Sketch of proof

Lemma. The theorem is true if $\langle W|D^N|V^r\rangle\alpha^r(1-q)^r = [y^r]\frac{\langle W|(D+yA)^N|V\rangle}{\langle W|A^r|V\rangle}$.

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Proof. Matrix Ansatz

"Guess and check"

Proposition

$$\frac{\langle W|A^r\mathsf{d}^{N-r}|V\rangle}{\langle W|A^r|V\rangle} = \frac{\sum_{i=0}^{N-r}(-1)^i \begin{bmatrix} N-r \\ i \end{bmatrix}_q q^{\binom{i}{2}}(bdq^r)^i B_{N-r-i}(b,d,q)B_i(a,c,1/q)}{\prod_{i=0}^{N-r-1}(1-abcdq^{2r+i})}$$

$$B_m(b,d,q) = \left(\sum_{j=0}^{m} \begin{bmatrix} m \\ j \end{bmatrix}_q b^j d^{m-j}\right)$$

Enumeration formula

Theorem. [Stanton 15]

$$Z_{N,r} = \sum_{k=0}^{N} \sum_{j=0}^{k} \frac{F_{k,r} q^k q^{-j^2} a^{-2j}}{(q,q^{1-2j}/a^2;q)_j (q,a^2 q^{1+2j};q)_{k-j}} (1 + aq^j + 1/(aq^j))^N / 2^N$$

$$F_{k,r} = (-a)^r \begin{bmatrix} k \\ r \end{bmatrix}_q \frac{(abq^r,acq^r,adq^r,q)_{k-r}}{(abcdq^{2r},q)_{k-r}} \frac{(q;q)_r}{(abcd;q)_{2r}} (ab, ac, ad, bc, bd, cd; q)_r q^{\binom{r}{2}}$$

$$a = \frac{1-q-\alpha+\gamma+\sqrt{(1-q-\alpha+\gamma)^2+4\alpha\gamma}}{2\alpha},\ b = \frac{1-q-\beta+\delta+\sqrt{(1-q-\beta+\delta)^2+4\beta\delta}}{2\beta}$$

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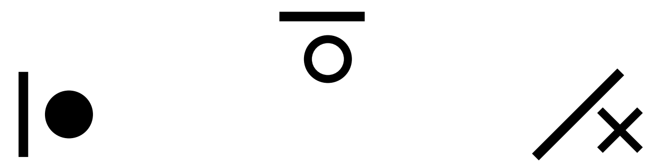
Remark. $Z_{N,r} \prod_{i=r}^{N-1} (\alpha\beta - \gamma\delta q^i)$ is a polynomial with positive coefficients in $\alpha, \beta, \gamma, \delta$ and q with $4^{N-r}(N-r)!\binom{N}{r}^2$ terms

Can we extract the combinatorics of the two species ASEP?

Triangular alternative tableaux

$q = 0$ [Mandelshtam 14]

$\gamma = \delta = 0$ [Viennot, Mandelshtam 2015]



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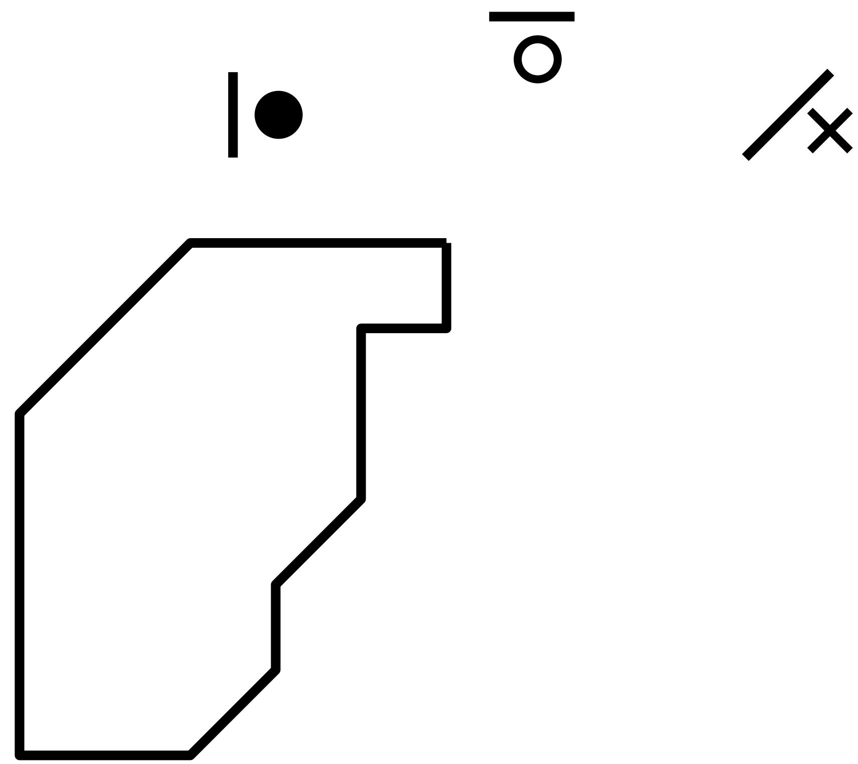

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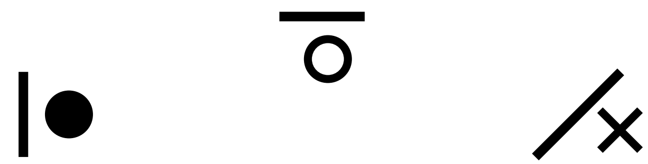

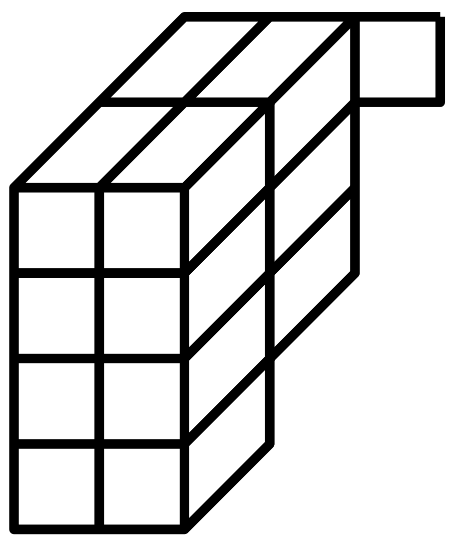

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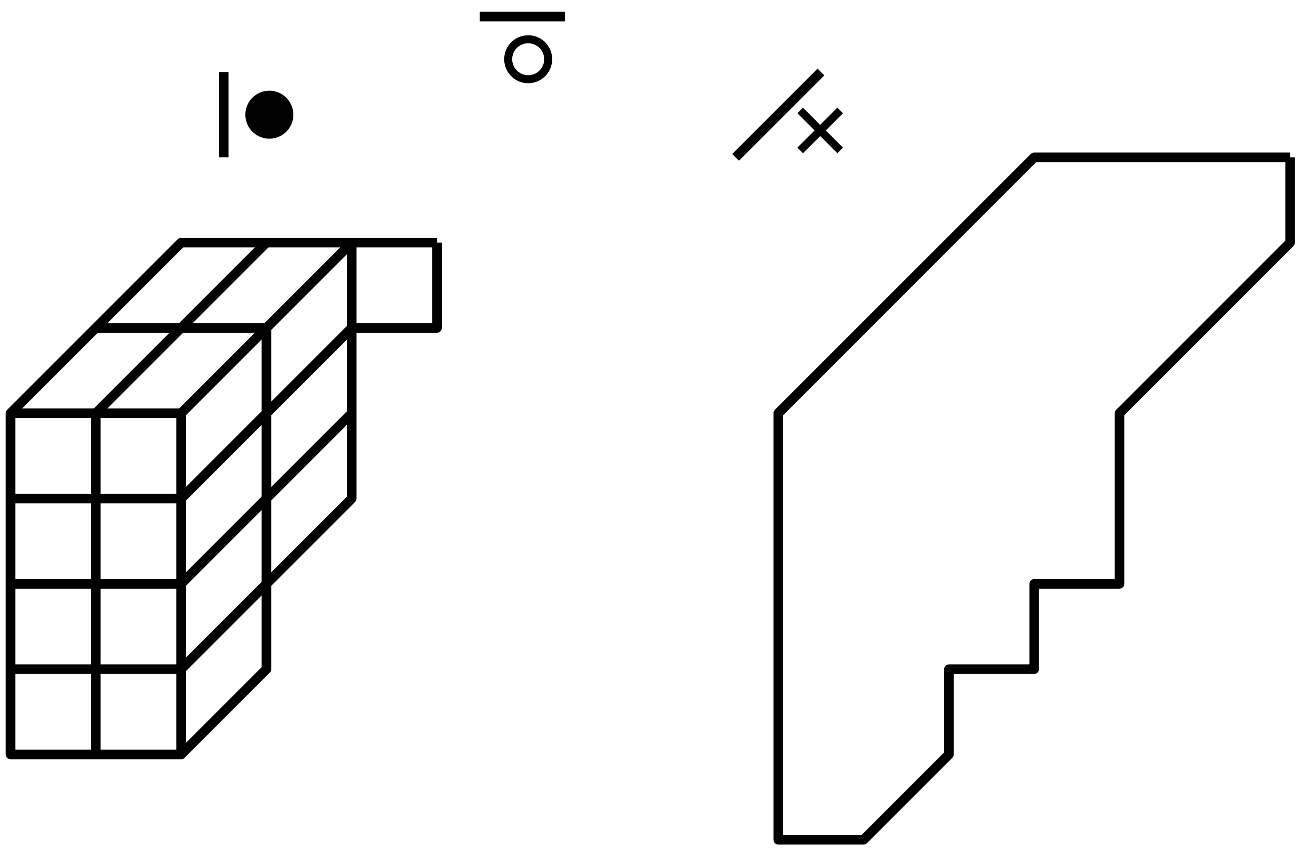

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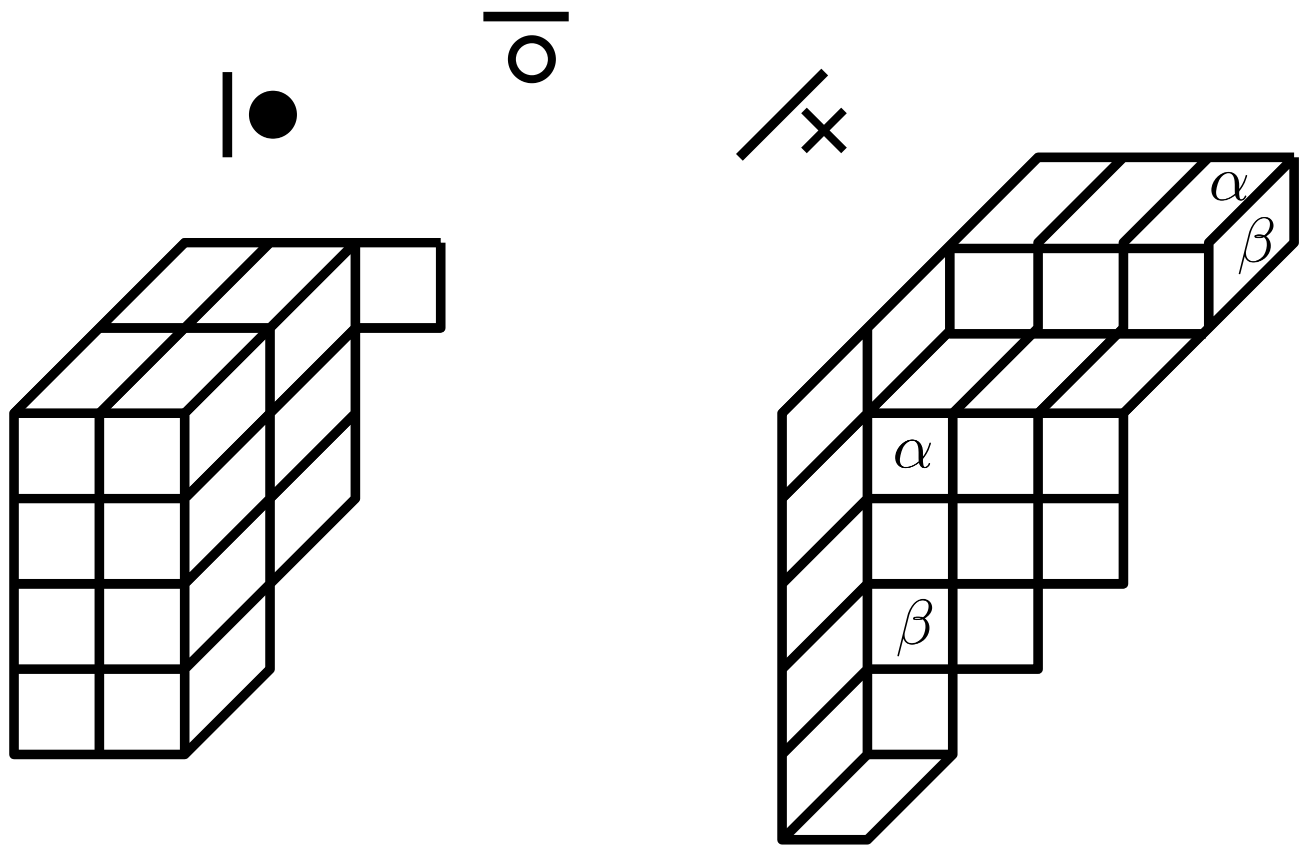

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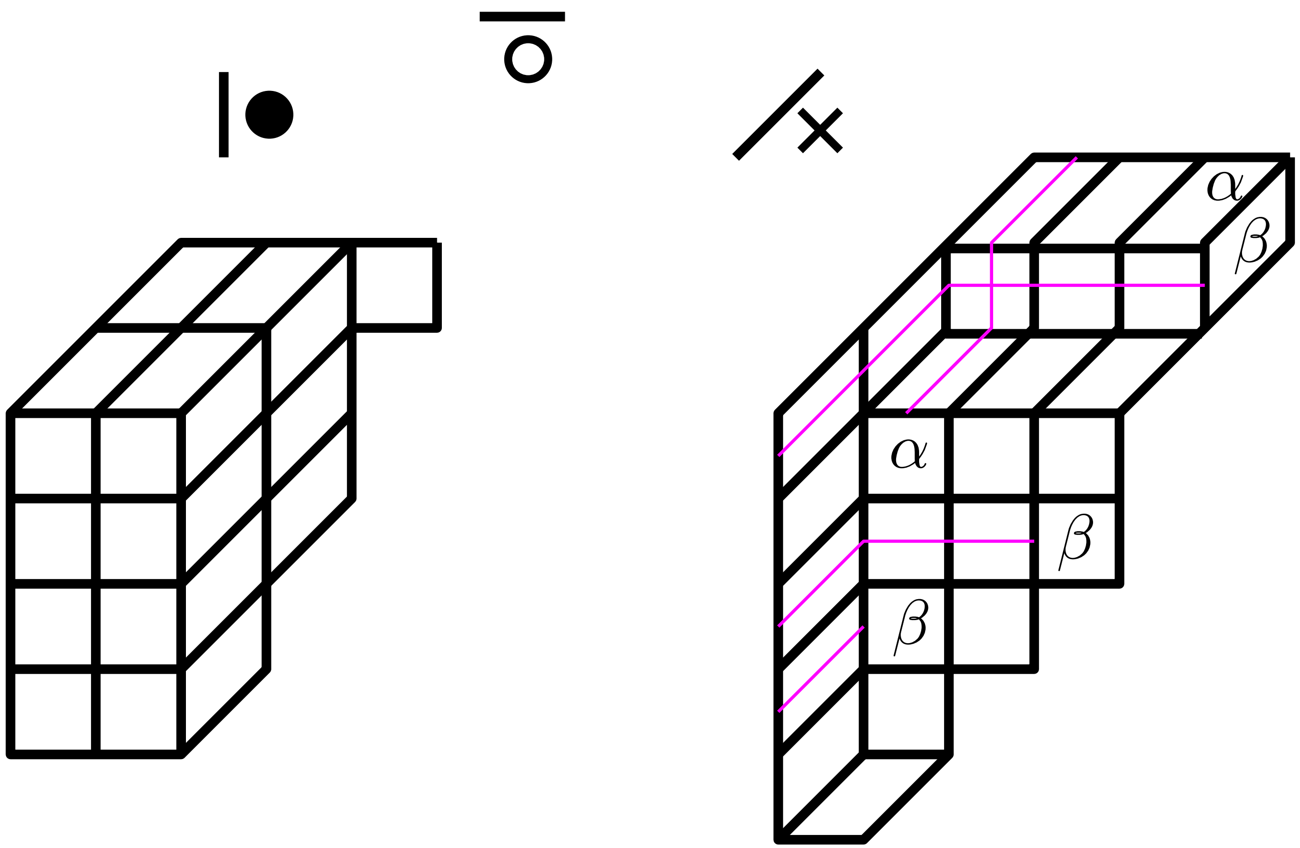

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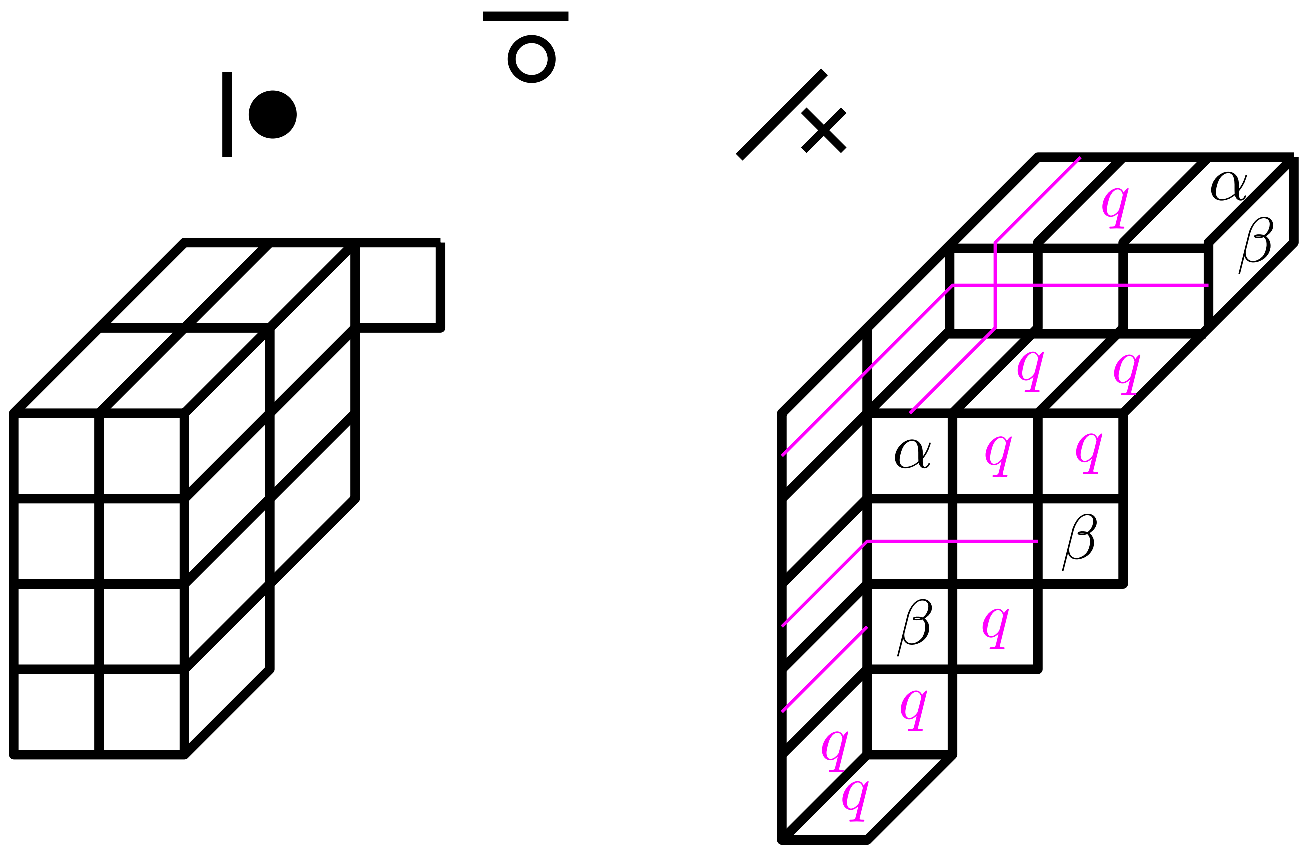

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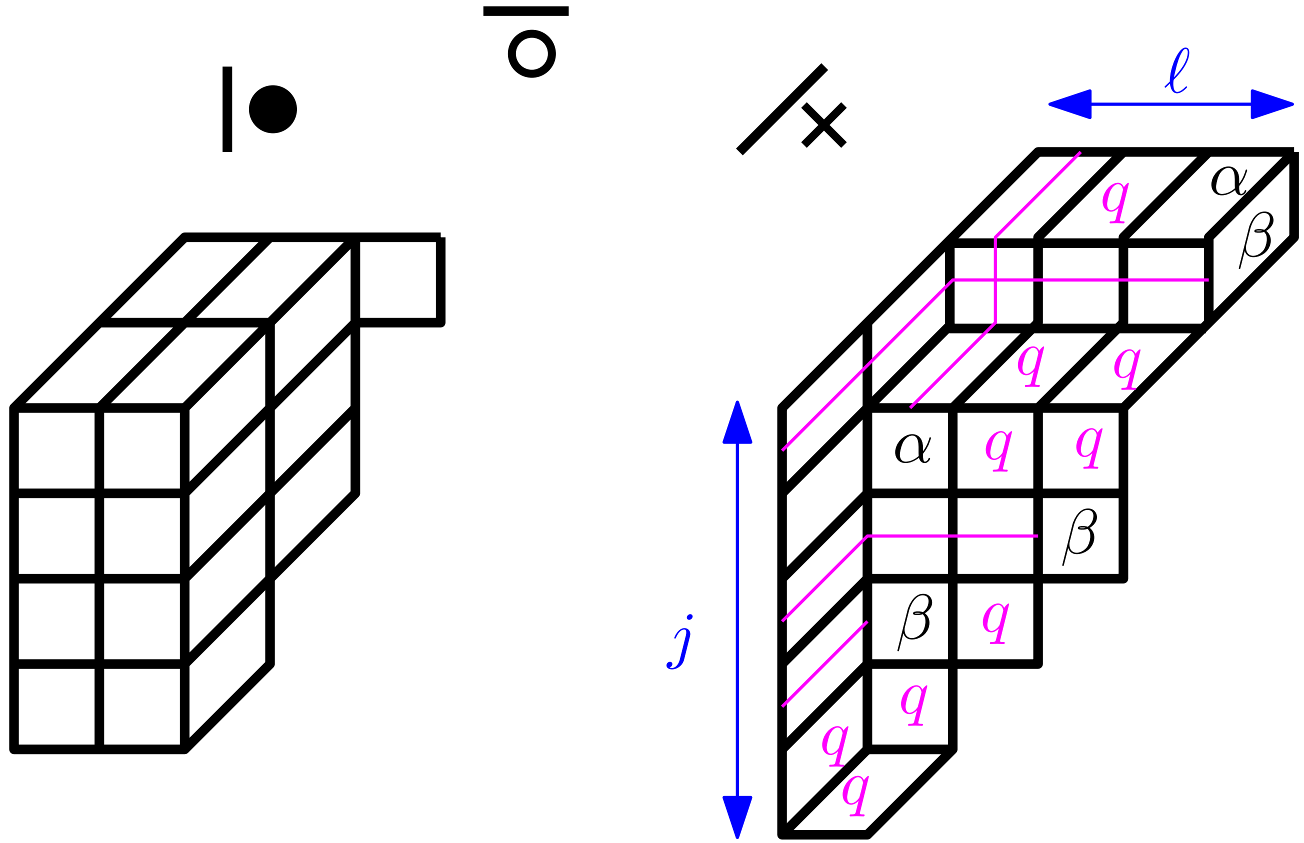

Weight$= \alpha^j \beta^\ell \prod \text{entries}$

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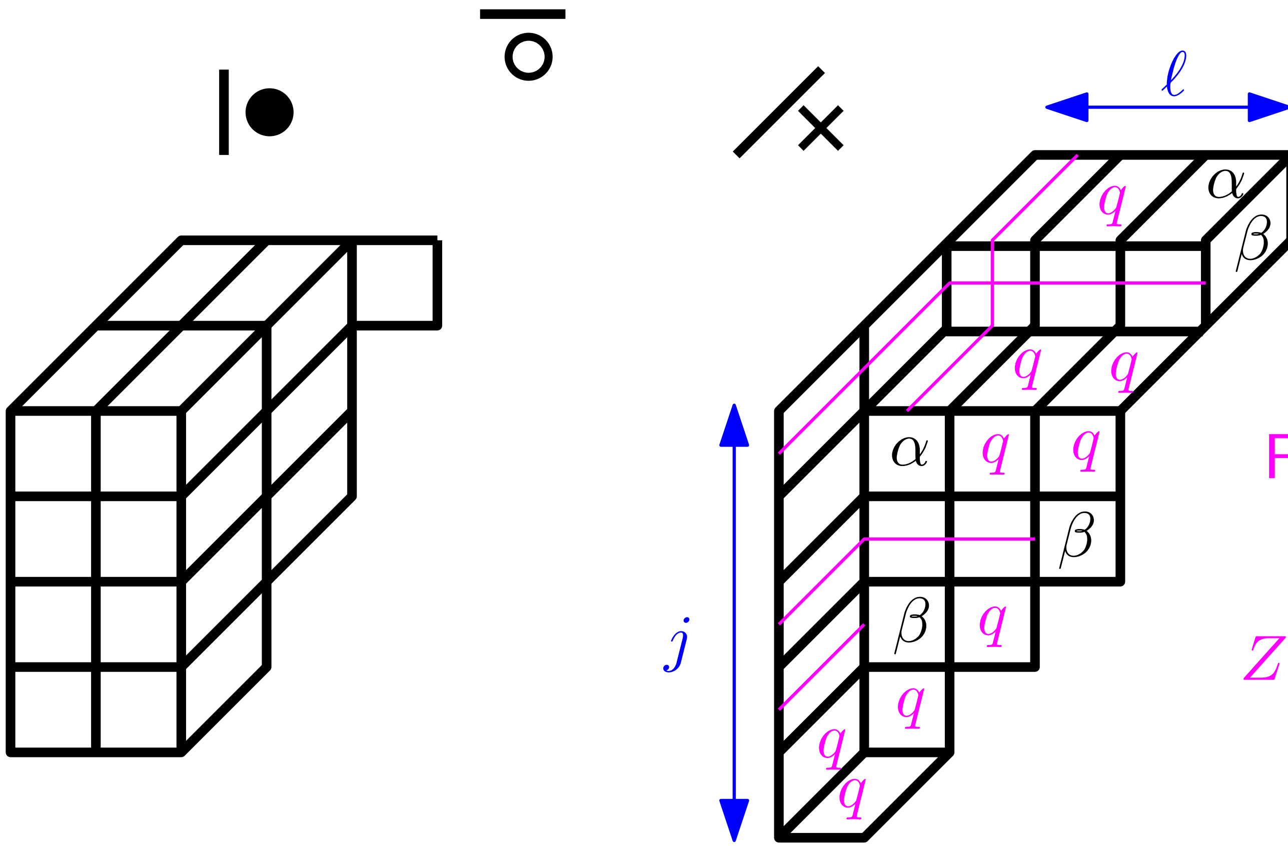

Weight$= \alpha^j \beta^\ell \prod$ entries

Fix a tiling t

$Z(\tau, t) = \sum_T \text{weight}(T)$

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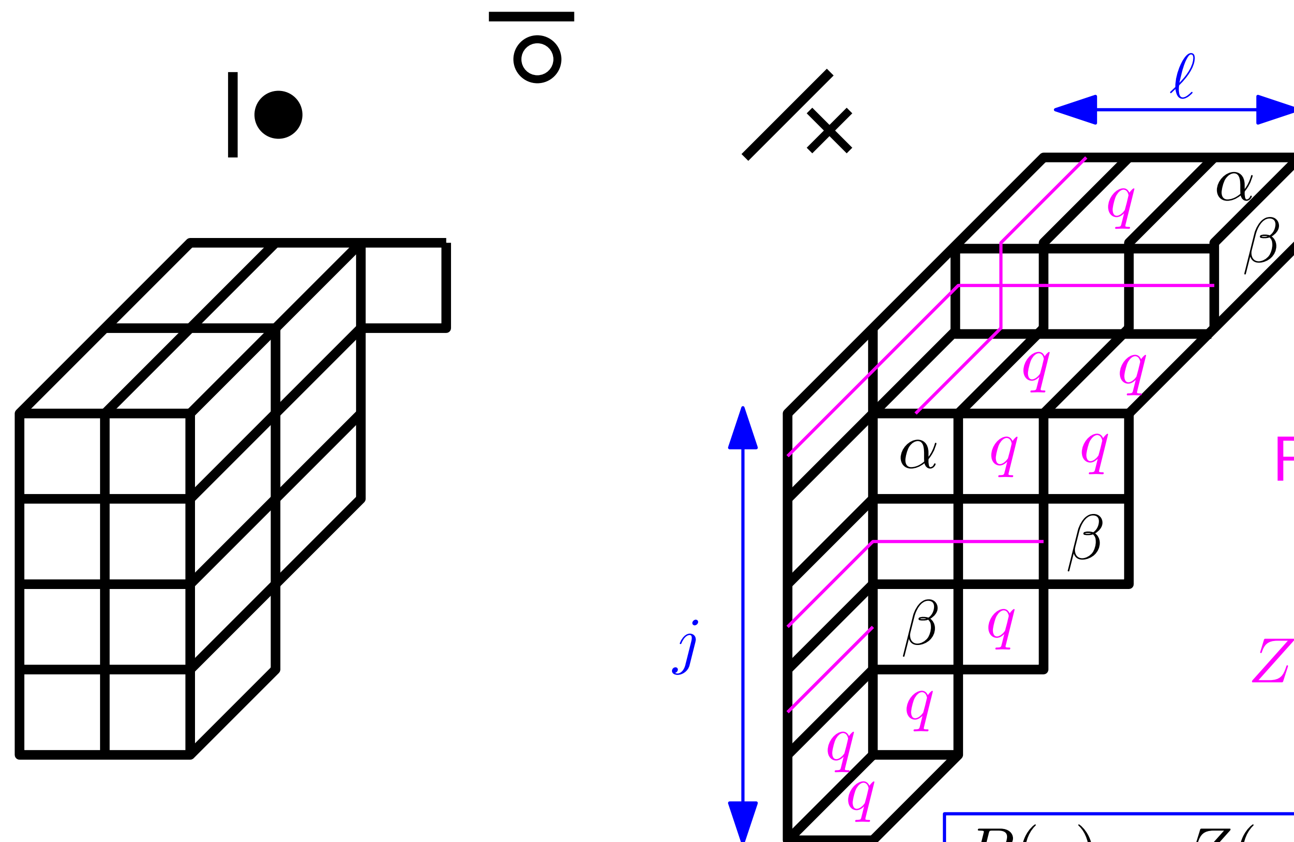

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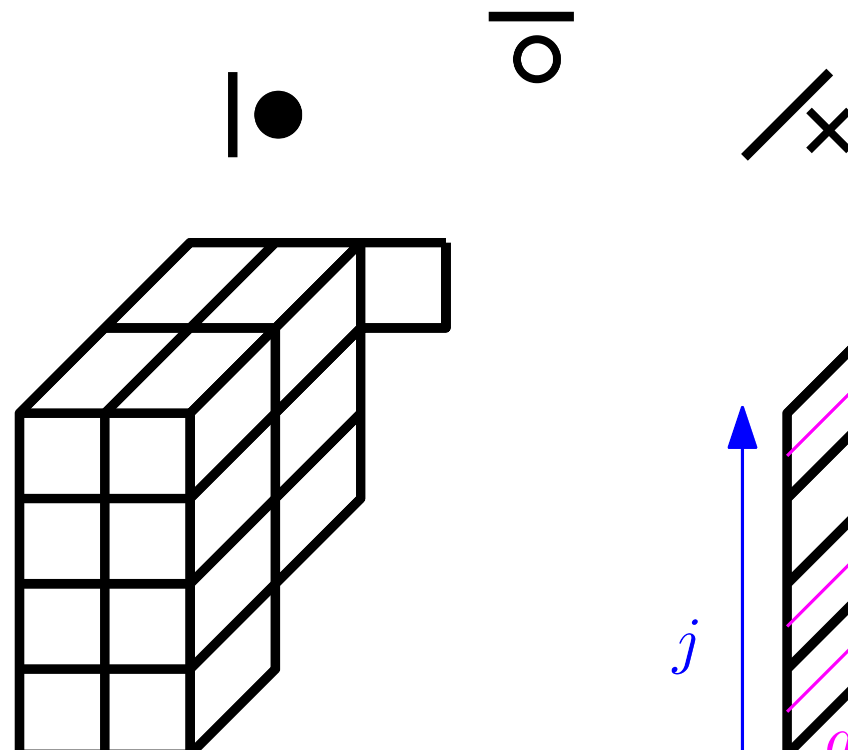

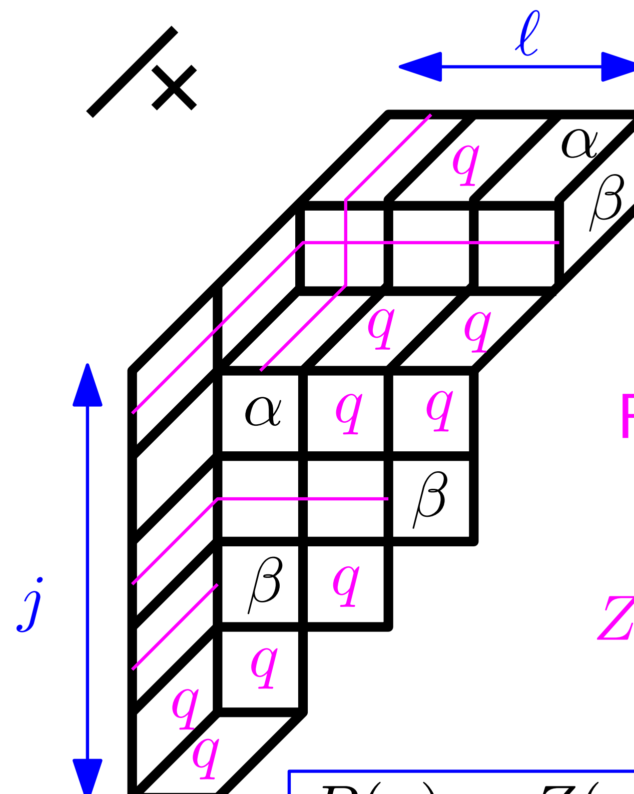

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For t and t' tilings, $Z(\tau, t) = Z(\tau, t')$

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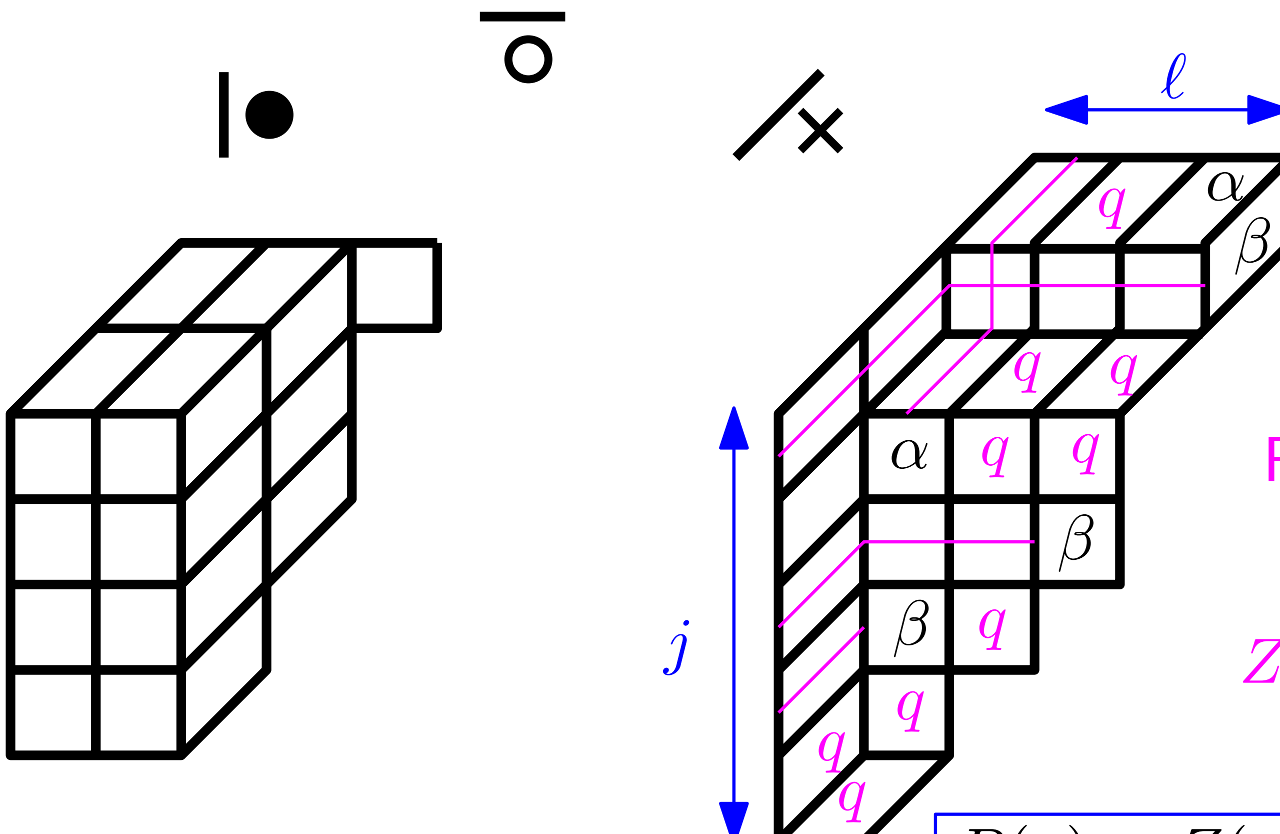

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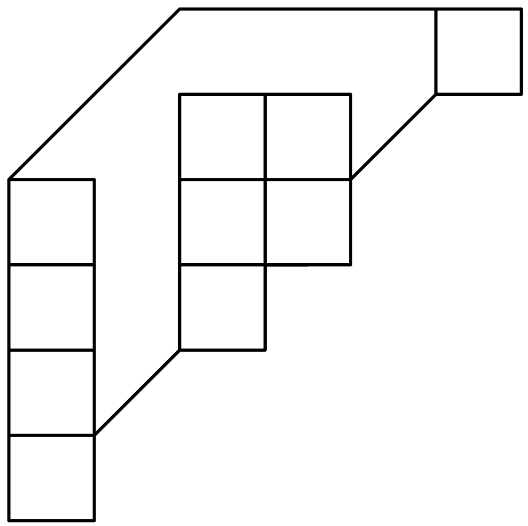
$\binom{N}{r} \frac{(N+1)!}{(r+1)!}$ tableaux

Triangular staircase tableaux [C., Mandelshtam, Williams 15]



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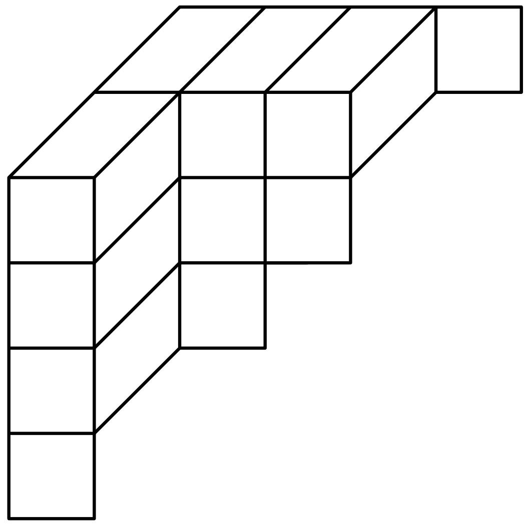

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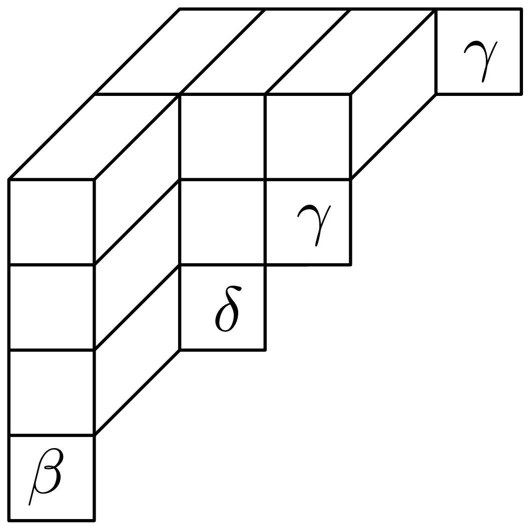

$\uparrow \alpha, \ \gamma$

$\leftarrow \beta, \ \delta$

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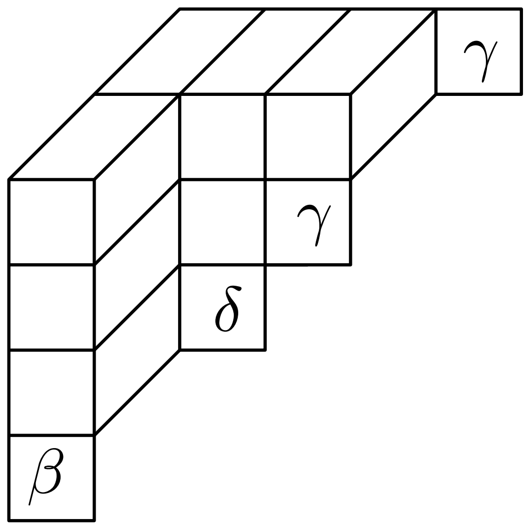
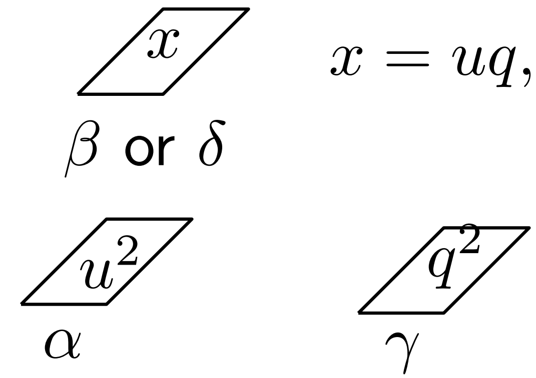
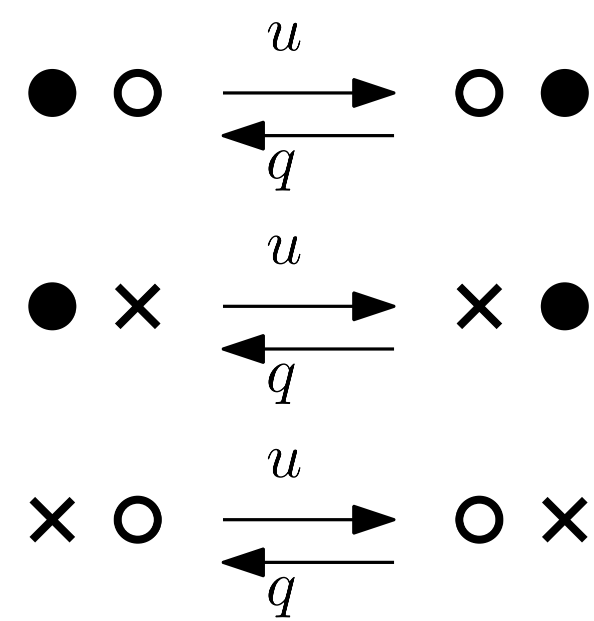




$x = uq, \alpha u$ or γq



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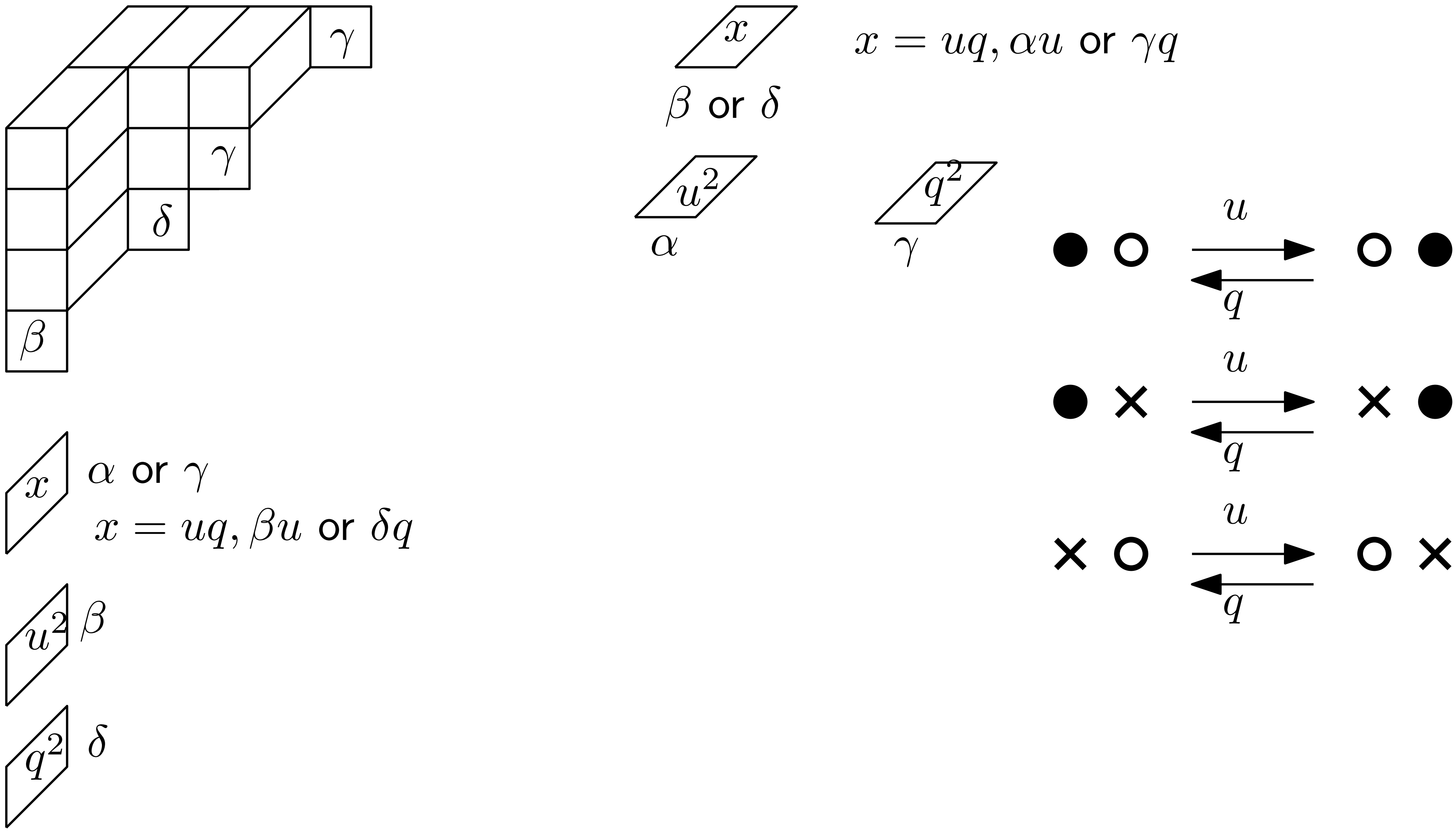
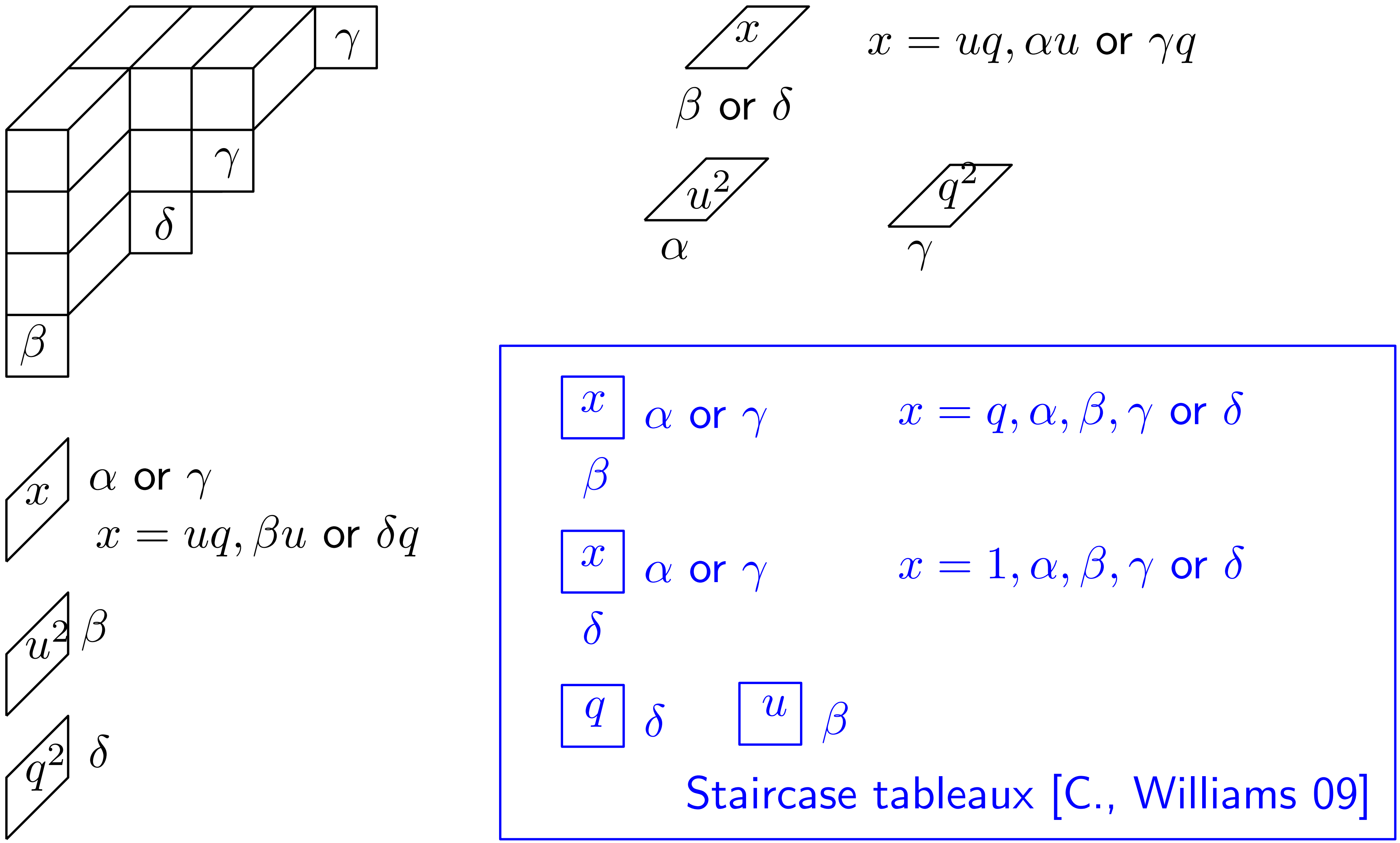


Triangular staircase tableaux [C., Mandelshtam, Williams 15]



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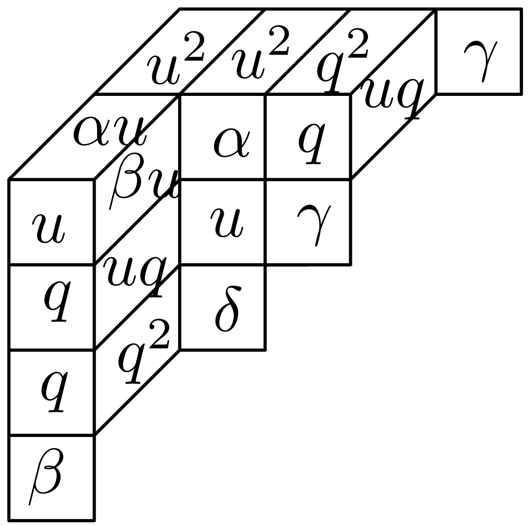
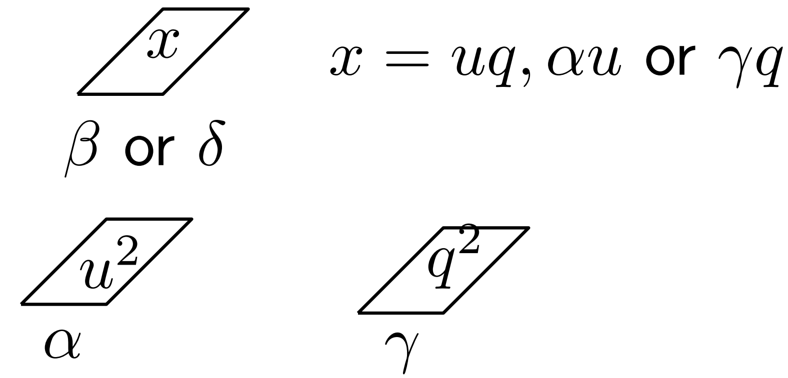
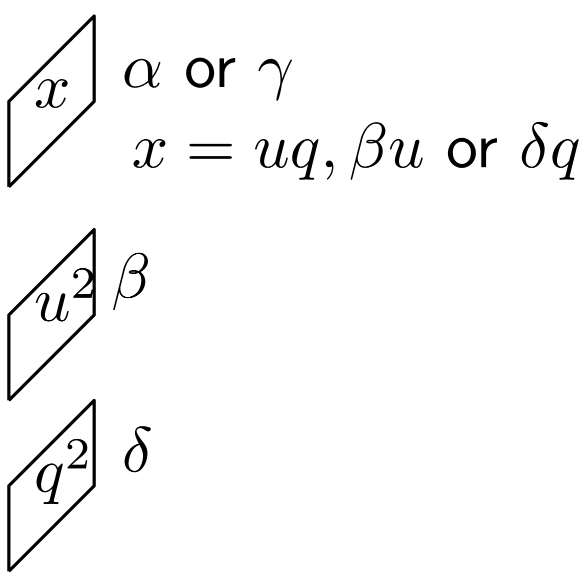
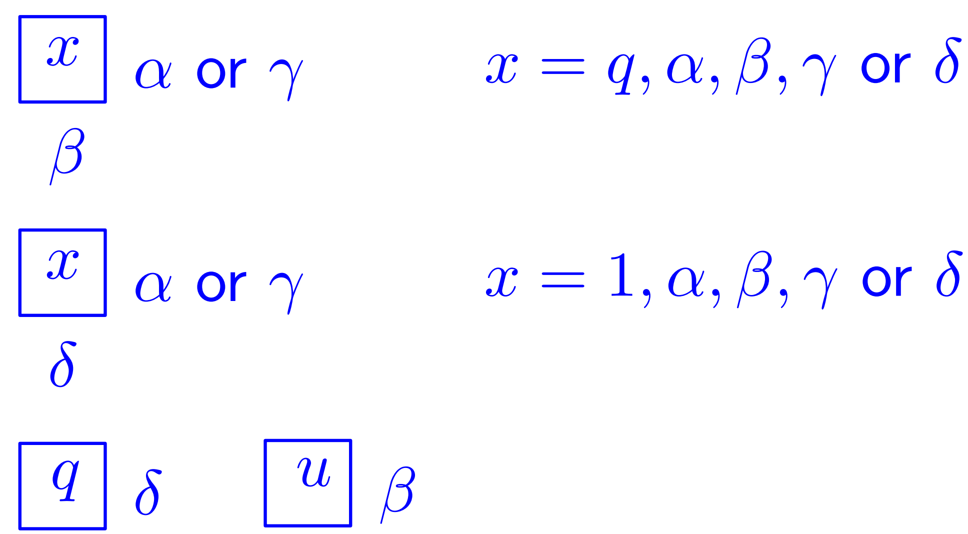


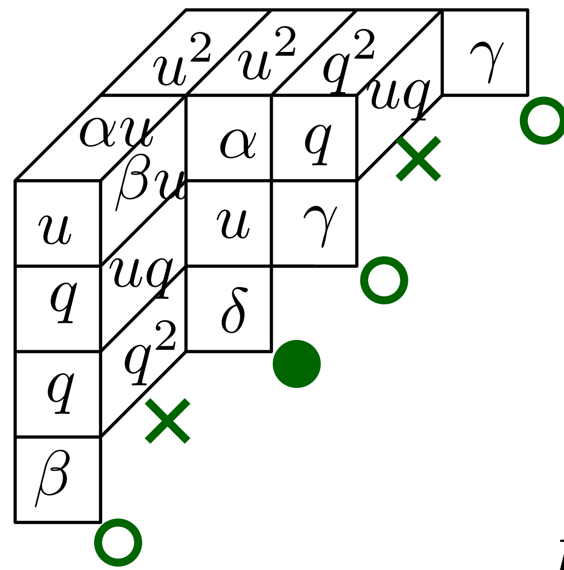






Staircase tableaux [C., Williams 09]

Type



$$Z_\tau = \sum_{T \text{ type } \tau} W(T)$$

$$Z_n(\alpha, \beta, \gamma, \delta, q, u) = \sum_\tau Z_\tau$$

$$P(\tau) = Z_\tau / Z_n$$

$$Z_{N,r}(\alpha, \beta, \gamma, \delta, 1, 1) = \binom{N}{r} \prod_{i=r}^{N-1} ((\alpha+\gamma)(\beta+\delta)i + \alpha + \beta + \gamma + \delta)$$

$4^{N-r}(N-r)!\binom{N}{r}^2$ tableaux

Bijective proof?

More to do?

- Links with Affine Hecke algebras?

  More species: more tiles [Mandelshtam 15+]

- 2-species where both types can enter and exit [Mallick et al 14]

- How to prove the general conjecture?

Conj. K_λ is a polynomial in $\alpha, \beta, \gamma, \delta, q$ with non-negative coefficients

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