Contents 1987

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Quillen’s own index for January 11 - March 2, 1987,

January 29: $K$-theory cup product using Clifford addition and multiplication.

January 31, February 1: The Gysin homomorphism in $K$-theory - integration along a fibre. Properties of $L = \begin{pmatrix} 0 & \partial_x - f \\ \partial_x + f & 0 \end{pmatrix}$.

February 3, 5: Quillen’s commentary and related thoughts on the paper by Goddard and Olive: Kac- Moody and Virago Algebras in relation to Quantum Mechanics.

February 7: Transmission lines and strings. Singular Dirac operators and resolvant.

February 8, 9: Reformulation of transmission line equations in terms of scattering data.

February 10: Using Clifford algebras to understand the normal ordering process.


February 12: Vertex operators.

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February 13: Clifford algebra of $L^2(S^1)$ with the quadratic form $\frac{1}{2} \int f^2 \frac{dx}{2\pi} = \frac{1}{2} \Sigma f_k f_{-k}$ where $f = \Sigma f_k e^{ikx}$, and fermion fields. Virasoro algebras. Commutation relations.


February 15, 16, 17: List of various approaches towards the existence of the resolvant $\frac{1}{\lambda - z}$ for $z = h\gamma^\mu \partial_\mu + \sigma X$ where $X = \frac{g-1}{g+1}$. Formulas for Virasoro algebras and $\frac{1}{2}$-densities.

February 18: Discussion of quantum field theory on Riemann surfaces.

February 19: More Virasoro calculations.

February 22: Ising model.

February 23: Clifford algebra calculations.

February 25, 26: Dirac operator over a circle with super-connection coefficients. Doubling.

February 27: Notes for Roe on $h\gamma^\mu \partial_\mu + \sigma X$ followed by a discussion of the same operator.

February 28: Skew-adjoint operators and subspaces. Approaches to constructing $U_g$ where $g : M \rightarrow U(V)$ and $U_g$ acts on $L^2(M, S \oplus V)$.

March 1: Finish example $Q_X = h\gamma^1 \partial_x + \gamma^2(-ia)$ where $a \in \mathbb{R}$.

March 2: More notes for Roe on Von Neumann’s approach to unbounded skew-adjoint operators and $g \mapsto U_g$ as for Feb. 28.

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Quillen’s own index for March 4 - April 21.

March 4: Review of various facts about symplectic structures.

March 10: Conformal fields theory on Riemann surfaces.

March 11: Real fermion fields on the torus.

March 12: The cohomology of a holomorphic vector bundle using meromorphic and adelic ideas.

March 13: Witten’s QFT on a Riemann surface.
March 14: Complex Fermi field. Imaginary time evolution.
March 15, 17: On a vector space of operators associated to rational functions of $z$.
March 18: Spinors over $\mathbb{P}^1$. Fock spaces attached to a Riemann surface.
March 19, 20: Fock space attached to a point.
March 21: On a Riemann surface with a single boundary component.
March 24: Review of facts about Clifford algebras.
March 26: Structure of arguments used for the Ising model.
March 27: The Ising model.
March 28, 29: Fermionization in the Ising model.

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April 8: Getzler and Toeplitz operators.
April 9: Pairing the Dirac operator on the circle with a loop $g : S^1 \to U(V)$ using Kasparov $KK$-theory cup product.
April 10, 11: Yesterday’s pairing leads to a Dirac operator on the torus.
April 12: Kasparov Hilbert $C^*$-module point of view applied to pairing a Dirac operator on the circle and a loop $g : S^1 \to U(V)$.
April 13 - 17: Pseudo-differential operators on $S^1$.
April 18: Notes on the map $S^1 \times S^1 \to S^2$ related to cup product in $K$-theory, and non-smoothness.
April 19: Smooth version of the cup product for $U(V) \times U(W) \to Gr(\mathbb{C}^2 \otimes V \otimes W)$. Notes for J. Roe on a Kasparov type construction for pairing an odd $K$-cohomology class of $S^1$ with the fundamental $K$-homology class.
April 21: Counter-example encountered in writing the notes for J. Roe.

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April 27: Torus example. Limiting behaviour of the graph of $\overline{\partial}$.
April 29: Graph of $\partial_x + bx$ for large $b$. Proofs of Witten’s result.
April 30, May 1: Review of problem of coupling a loop $g : S^1 \to U(V)$ to Dirac $\partial_x$ on $S^1$.
May 2: Approximating $\partial_x \pm x$ on $\mathbb{R}$ by an operator on $(-\delta, \delta)$.
May 3: Analysis of the ‘blip’ operator given by a loop $g : S^1 \to (U(1)$ of winding number 1, concentrated near 1.
May 4, 5: Construction of a certain pseudo-differential operator of index 1 on $S^1$.
May 6: Coupling the Dirac operator on $S^1$ to a unitary loop. Loring’s projector.
May 7: Papers on bosonization on a Riemann surface by Bost-Nelson, Alvarez, Gomez and Vafa.
May 8: Parametrix methods applied to the coupling problem.
May 9: Some analysis of elliptic operators.
May 10: Invertibility of $s - D$ where $D = \begin{pmatrix} 0 & \partial_x - q \\ \partial_x + q & 0 \end{pmatrix}$ for $|\text{Re}(s)|$ large.

May 11: More on pairing the Dirac on a circle with a loop.

May 12: On Weil’s Acta paper about an adelic version of the $\theta$-function and a proof of quadratic reciprocity associated to $K_2(\mathbb{Q})$.

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May 13: On Weil’s Acta paper.

May 14: Analysis of a real scalar field $\phi$ on $S^1$ with Hamiltonian $\int \frac{1}{2} \phi^2 dx + \int \frac{1}{2} \phi (\partial_x^2 + m^2) dx$.

May 26: Bosonization and Graeme Segal’s ideas on QFT for holomorphic functions.

May 28: Log-harmonic functions.

May 29 - June 3: Work on bosonization.

June 5, 6: The annihilator of $\Gamma(X, \theta)$ is $\int \Gamma(X, \Omega) (\text{Kronheimer})$ and related results. A line in the Fock space of $C^\infty(S^1)/\mathbb{C}$ associated to a Riemann surface with boundary circle $S$.

June 7: Symplectic vector spaces and self-dual lattices.

June 8, 9, 10: On a real symplectic vector space and its Heisenberg representation.

June 15, 26: On the cohomology of a Riemann surface with $g$ handles and $r$ boundary circles. More on QFT.

June 17: Further discussion of a real symplectic vector space and its Heisenberg representation.

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June 18, 19: Functions and 1-forms on a Riemann surface with boundary.

June 20, 21: Cohomology of the double of a Riemann surface with boundary. Riemann’s conditions for a closed Riemann surface.

June 22 -25: Calculations for a complex vector space with a Hermitian form or a symplectic form.

June 26 - July 2: Quasi-free states for a real symplectic vector space with a skew form $S(x, y)$ and Weyl algebra with $[x, y] = i S(x, y)$.

July 4: On $\theta$-functions.

July 5: The link between Riemann type $\theta$-functions and intrinsic $\theta$-functions.

July 6: Note on a function satisfying a quasi-periodicity property.

July 7, 8, 9: The category of real finite dimensional vector spaces and complex symplectic morphisms.

July 10: An intrinsic $\theta$-function on a complex vector space with a Hermitian inner product and a lattice.


July 13, 14: An account of $\theta$-functions.

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July 15, 20: Coupling super-connections with Dirac operators.

July 21, 22, 23: On the equation $(p^x F p^x + iq) \phi = f$ with $p = g^{- \frac{1}{2}} (2q + 1), q = g^{- \frac{1}{2}} (2q - 1)$.
July 26: The operator $g^{\frac{3}{2}}(pF+iq) = \left(\frac{g+1}{2}\right)F + \left(\frac{g-1}{2}\right) = gP_+ - P_-$ is an elliptic pseudo-differential operator of order 0.

July 31, August 1, 2: Differential operator case, looking at the subspace $\text{Im} \left( \begin{array}{c} r \\ p\partial_x + q \end{array} \right)$ and calculations with the Green’s function.

August 2, 3: On the ordinary differential equation $u - pu' = f$.

August 5: Transmission lines.

August 6: Putting in Planck’s constant.

August 7: On the Dirac operator in the form

$$\begin{pmatrix}
0 & h\left(\frac{p}{r}\right)^{\frac{1}{2}} \partial_x \left(\frac{p}{r}\right)^{\frac{1}{2}} - \frac{q}{r} \\
\left(\frac{p}{r}\right)^{\frac{1}{2}} \partial_x \left(\frac{p}{r}\right)^{\frac{1}{2}} + \frac{q}{r} & 0
\end{pmatrix}.$$

August 8: Parametrix methods.

August 13: Note on defining a subspace of $L^2(\mathbb{R})^2$ by $\text{Im} \left( \begin{array}{c} r \\ L \end{array} \right)$ where $L = p\partial_x + q$.

August 14: On minimal and maximal closures of densely defined operator.

August 15: On the equation $D_x P(x, y) = \delta(x,y) - K(x,y)$ where $D$ is a differential operator, $P$ a distribution smooth off the diagonal and $K$ a smooth function.

August 16: On Schur’s estimate.

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August 23: Friedrich’s mollifier methods.

August 25 - September 5: Coupling $\partial_x$ with an even $K$-class.

September 6, 7: Review of problem of given a unitary matrix function $g$ to find a connection $d_B$ such that if $\rho_1$ and $\rho$ are suitable functions on $S^1$ vanishing at $-1$ then $\rho_1(g)^{-1}(d + B)\rho(g)$ is smooth.

September 8: Alterative approaches to constructing a pairing between $\partial_x$ and $g$.

September 9, 10: Cup product map $\mathbb{R} \cup \{\infty\} \times \mathbb{R} \cup \{\infty\} \rightarrow \mathbb{C} \cup \{\infty\}$ given by $(x, y) \mapsto x + iy$. Smoothing the singularity.

September 12: Non-abelian case of associating to a loop an index -1 pseudo-differential operator of the form $r(h\partial_x + f)^{-1}$.

September 14 - 17: Arranging for $r \sum_{i=0}^{n-1} f_0^{n-1-i}(\partial_x + f_1)(f_0')$ to be smooth.

September 18: From a $2 \times 2$ hermitian matrix $A$, construct a 1-form $rf$ such that $rdA + rfA - Arf$ is of the form $Ts(A)$ with $s(x)$ vanishing to large order at $x = 0$.

September 19: Application of the $2 \times 2$ matrix calculation.

September 26, 27: Calculation for when $g$ has a constant number of eigen values equal to -1.

September 28, 29: Review of progress.

September 30: Calculations for $X = \begin{pmatrix} X' & 0 \\ 0 & X'' \end{pmatrix}$ where $X''$ is invertible and $\partial + \sigma X = \partial \begin{pmatrix} \partial' + \sigma X' & i\ast\partial_j j^* \partial_i \\ j^* \partial_i & \partial'' + \sigma X'' \end{pmatrix}$ forming an inverse.

October 1: Review of known methods for coupling $\partial_x$ to an arbitary loop $g$. 
October 2: On the pseudo-differential operator with complete symbol equal to \( g \) for \( \xi > 0 \) and 1 for \( \xi < 0 \) where \( g : S^1 \to U(V) \) and similarity with the Bott map \( S^1 \times U(V) \to \text{Gr} (\mathbb{C} \otimes V) \).

October 3: Interpreting maps \( \mathbb{R} \cup \{ \infty \} \times (\mathbb{R}/\mathbb{Z}) \to \text{Gr} (\mathbb{C}^2 \otimes V) \) as points in the restricted Grassmannian.

October 5: Connes groupoid theory and turning a map \( T^*(S^1) \to \text{Gr} (\mathbb{C}^2 \otimes V) \) into an operator.

October 6: Quantizing maps \( T^*(S^1) \to \mathbb{P}^1 \) which represent the canonical \( K \)-class.

October 7: Deformation of the algebra of functions on \( T^*(S^1) \).

October 8: On using \( K \)-theory exact sequences. Understanding the map \( K_1(A/I) \to K_0(I) \) in algebraic \( K \)-theory.

October 9: Toeplitz operators.

October 10: More on \( K_1(A/I) \to K_0(I) \).

October 11: The index of a projector over the algebra of the deformation.

October 13: Review of the algebras \( A_h = C^\infty (S^1) \otimes S(\mathbb{R}) \) with twisted product \( e^{-ix}f(p)e^{ix} = f(p+h) \), and \( \mathscr{A} = C^\infty (S^1) \otimes \mathcal{S}(\mathbb{R} \times [0,1]) \).

October 15: Ideas from yesterday’s lecture on vector bundles over \( X/Y \).

October 16: More calculations in the twisted algebra including as calculation of the index of an involution.

October 17: Correction of an error on October 11.

October 19: More on an involution \( F(h,x,p) = \sum_{n=0}^\infty h^n F_n(x,p) \).

October 20: More on \( A_h \) and \( \mathscr{A} \).

October 22: On differential graded algebras. Representation of \( \mathscr{A} \).

October 24: On the index of a pair of idempotents whose difference is compact.

October 25: Multiplier algebra.

October 27: On \( \mathscr{A} = q\mathbb{C} = \text{Ker} \{ \mathbb{C}e * \mathbb{C}e' \to \mathbb{C}e \} = \text{Ker} \{ \mathbb{C}[\mathbb{Z}/2 * \mathbb{Z}/2] \to \mathbb{C}(\mathbb{Z}/2) \} \). Trace on \( \mathscr{A} \) in \( K_0(\mathscr{A}) \).

October 28: Fredholm modules.

October 29: More on the algebra of \( f(h,x,p) \).

October 30, 31: Formal problem: to compute the index associated to a trace on \( \tilde{\mathscr{A}} \) and an involution on \( \mathscr{A}_0 \) where \( \mathscr{A}_0 = \text{Schwartz functions on } T^*(M) \) and \( \tilde{\mathscr{A}} = \mathbb{C}[[h]] \), deformed for the tangent groupoid.

November 1: Generalizing the \( n = 1 \) case where the character form is \( \text{tr}(F_0(dF_0)/2) \) to higher \( n \). Non-commutative differential calculus. \( K_0(A) \to K_0(A/I) \) if \( I \) is nilpotent. Calculating \( K_0 \) for \( \mathbb{C}[\mathbb{Z}/2 * \mathbb{Z}/2] \) and the \( C^* \) algebra \( \mathbb{C}(T) \times (\mathbb{Z}/2) \).

November 2: Infinitesimal ways of locating elements of \( K_0(\mathscr{A}) \). \( KK \)-theory.

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November 3, 5: To construct a mechanism for saying that a trace on \( \tilde{A} \) (\( A \) extended by a nilpotent ideal) becomes equivalent to a cyclic cocycle on \( A \).

November 6, 7: Square zero extensions. Non-commutative \( \Omega^1 \). \( HC_1(A) \).

November 10: Cyclic homology for a vector space with multiplication.
November 11: Non-unital algebras over a field of characteristic 0.

November 12: Super-algebra context to resolve two approaches to cyclic homology.

November 13: Studying extensions.

November 14, 15: To define a canonical map $HC_2(A) \to Q/[Q,Q]$ associated to a square zero extension of $A$: $0 \to M \to Q \to A \to 0$.

November 16: A process for converting an extension into a Fredholm module.


November 19, 20: More on cyclic homology, extensions and traces. Deformation theory of $HC_2(A)$.

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November 21: On a semi-direct product $Q = A \oplus M$ where $A = T(V)$ is free unital and $M$ is a bimodule over $A$.

November 22: On the cyclic homology of $A \oplus M$ of a super-algebra $A$ and an $A$ super-bimodule $M$ with $M^2 = 0$.

November 23: There is no canonical map $HC_3(A) \to HC_1(Q)$ where $Q$ is a square zero extension of $A$.

November 24: An $A$-bimodule over $T_A(M)$.

November 25: Goodwillie’s proof that if $Q = A \oplus M$ then Connes($Q$) = Connes($A$) $\bigoplus_{k \geq 1} (\tilde{M} \otimes A)^k / (1 - t)[k - 1]$.

November 26: The spectral sequence in cyclic homology for the extension $I \to P \to A$.

November 27: Calculation of $d' : E_{k+1,k} \to E_{k,k}$.

November 30: Exact sequences associated with $P = T(V)$, $P/I = A$.

December 1, 3: Calculations with $T_P(I)$.

December 4: On a Riemann surface with $g$ handles and $r$ boundary circles. Harmonic functions and forms.

December 5: More calculations with $A = P/I$, $P$ free. Product structure on $H^*(A, A)$.

1987-14


December 6: Normalized acyclic Hochschild complexes.

December 7: Applications of Connes Complex functor.

December 8: Foundations of cyclic theory and an approach different from Connes. Comment on Waldhousen’s algebraic $K$-theory. Hochschild and cyclic homology of free group algebras.

December 9: On the cyclic homology of the algebra $A$ of Schwartz functions on $T^*(M)$ and the algebra $P$ of formal power series with coefficients in $A$ and with a twisted multiplication. Gauss-Manin connection.


December 12: Comments on progress in developing cyclic homology.

December 14: The reduced cyclic homology of $a = \mathbb{C}[G]$ where $G$ is a free group.

December 15, 16: Review and a short proof of Goodwillie’s theorem.
December 18: More on the Gauss-Manin connection.

December 23: On understanding the map $(L \otimes_P)^k \sigma[1] \to (I \otimes_P)^k \sigma^{-1}$.

December 24, 25: Relative cyclic homology of $A$ relative to $P$.

December 27: Cyclic homology in terms of cyclic tensor products.