Clay mathematical institute, Oxford

Random polymers and algebraic combinatorics

What an exactly solvable (log-gamma) model tells us on polymer localization ?

Francis Comets

Université Paris-Diderot

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Joint work with Vu-Lan Nguyen.

Purpose

Directed polymer model ≡ random walk in a random potential.
Weight of a path = exponent of the sum of the potential it has met on its way.
Competition:

$$\underbrace{\text{entropy of paths}}_{\text{diffusive behvior}} \longleftrightarrow \underbrace{\text{disorder strength}}_{\text{inhomogeneities of potential}}$$

Path is attracted by "favorable spots" with large potential values.

Constant potential $\Longrightarrow$ path spreads smoothly over distances O $(\sqrt{\text{length}})$.
But if potential has large variability,

- ☞ path localizes on a few corridors with width of order of unity.
- ☞ spreads at "abnormally" large distance (superdiffusivity)

Recently, explicitely slovable models of planar polymers (i.e. $(1+1)$-dim.) have been discovered and used to study the second effect typical of the KPZ universality class.

Here, we take the Log-Gamma polymer to get sharp results for localization.

Outline

The Model

Environment: $(\omega(x), x \in \mathbb{Z}^2)$ real, independent identically distributed r.v.'s

Polymer path: over up-right paths $\mathbf{x} = (x_t; 0 \leq t \leq n)$, $x_t - x_{t-1} = \mathbf{e_1}$ or $\mathbf{e_2}$.

Point-to-line polymer measure of a path of length n is

$$Q_n^\omega(\mathbf{x}) = \frac{1}{Z_n^\omega} \exp\left\{ \sum_{t=1}^{n} \omega(x_t) \right\}$$

with $Z_n = Z_n^\omega$ the sum over all up-right paths $\mathbf{x}$ starting at $x_0 = (0,0)$.

For a typical realization ω of the environment, what is the behavior of the polymer of large size $n \to \infty$? (asymptotics of the quenched law)

The KPZ universality class

Recent efforts on planar polymer models to understand physics predictions: KPZ universality class (Kardar, Parisi and Zhang 1986), see Corwin 2012's recent survey.

Non-gaussian scaling limits and statistics, characterized by a few exponents.

KPZ class covers large random matrices, last passage percolation, interacting particle systems (TASEP,. . .).

Biblio starting from Johansson (1998 $\rightarrow$)

A few explicitely solvable polymer models:

- KPZ equation (Quastel 2010, Hairer 2013),
- Brownian queues O'Connell-Yor 2003,
- log-gamma polymer Seppäläinen 2012.

Log-gamma polymer

Change notations:

$$Y_{i,j} = e^{\omega(i,j)}$$

so that $Z_n = \sum_{\mathbf{x}} \prod_{t=1}^{n} Y_{x_t}$.

Def.: Log-gamma polymer with parameter $\mu > 0$ (abbrev. $\mathrm{Gamma}(\mu)$)

$$Y_{i,j}^{-1} \sim \Gamma(\mu)^{-1} x^{\mu-1} e^{-x} \quad , \qquad x > 0.$$

Seppäläinen 2012

Log-gamma polymer is in KPZ class

Seppäläinen 2012 discovered the stationarity property of this model, which makes it explicitely solvable:

- ☞ Seppäläinen obtains the value of the free energy

$$n^{-1} \ln Z_n \xrightarrow{n\to\infty} -\Psi_0(\mu/2), \qquad \Psi_0 = \Gamma'/\Gamma$$

  and proves that the volume and wandering exponents for fluctuations are

$$\chi = 1/3, \qquad \xi = 2/3.$$

- ☞ Large deviations of the partition function Georgiou-Seppäläinen 2013
- ☞ GUE Tracy-Widom fluctuations for Z_n at scale $n^{1/3}$
  Borodin-Corwin-Remenik 2013: for small μ,

$$n^{-1/3}\left(\ln Z_n + n\Psi_0(\mu/2)\right) \xrightarrow{\text{law}} F_{GUE}$$

  the GUE Tracy-Widom distribution.

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  the GUE Tracy-Widom distribution.
- Explicit formula for the Laplace transform of the partition function at finite size Corwin-O'Connell-Seppäläinen-Zygouras 2014; integral formula which can be turned into a Fredholm determinant.

Outline

Localization

Can be measured by probability of the "favourite endpoint"
(= largest probability among endpoints)

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In large generality it is proved that the polymer is localized: Carmona-Hu 2002, Comets-Shiga-Yoshida 2003 proved, by (semi)martingale methods, that

$$\lim_n n^{-1} \ln Z_n < \lim_n n^{-1} \ln \mathbb{E} Z_n \quad \Longleftrightarrow \quad \liminf_n n^{-1} \sum_{k=1}^{n} \mathcal{I}_k \geq c_0 > 0.$$

The assumption holds for planar models (Comets-Vargas 2006, Lacoin 2010). Then,

$$\limsup_{n \to \infty} \mathcal{I}_n \geq c_0$$

a sharp contrat with $O(n^{-d/2})$ with $d = 1$ when ω =Cst.

General techniques

- Doob's decomposition relies on (conditional) first moment, which can be estimated by the replica overlap

$$\frac{1}{n}\sum_{t=1}^{n}\mathbf{1}_{\{x_t=\tilde{x}_t\}}\,, \qquad x,\tilde{x}\sim Q^{\omega}_{n-1}$$

independent except they share the same environment. The overlap can be compared to $\mathcal{I}_n$.

- In absence of moments, an alternative approach was developped by Vargas 2007 leading directly to to $\mathcal{I}_n$.

Our aim here

Our aim is to find sharp asymptotics for localization in the log-gamma model. Natural questions are:

- ✘ Where does the mass of the polymer measure stem from ?
- ✘ What are the regions which contribute the most ?
- ✘ Are they many ? How far are they ? Are wide are they ?

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✍ In some models, the above questions can be answered in the limit

$$\text{Temperature} \to 0 :$$

Complete localization when the overlap converges to 1.
C-Cranston 2013, C-Yoshida 2013.

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✍ All previous methods are indirect.
In the log-gamma model we have explicit computations.

Connections with Parabolic Anderson model and KPZ equation

The point-to-point partition function

$$Z_{m,n} = \sum_{\mathbf{x}:\mathbf{0}\mapsto(m,n)} \prod_{t=1}^{m+n} Y_{x_t},$$

is a solution of the parabolic Anderson equation

$$Z_{m,n} = e^{\omega(m,n)}[Z_{m-1,n} + Z_{m,n-1}]$$

or, equivalently,

$$Z_{m,n} - Z_{m-1,n-1} = [2 - e^{-\omega(m,n)}]Z_{m,n} + \big(Z_{m-1,n} + Z_{m,n-1} - Z_{m,n} - Z_{m-1,n-1}\big),$$

prod. Z with a noise

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a discrete version of Stochastic Heat Equation ($t = n+m$, $r = n-m$, $\mathcal{Z}(t,r) = Z_{n,m}$, and $\eta(t,r) = [\,\cdot\,]$ in the above)

$$\frac{\partial}{\partial t}\mathcal{Z}(t,r) = \eta(t,r)\mathcal{Z}(t,r) + \frac{\partial^2}{\partial r^2}\mathcal{Z}(t,r)$$

The Hopf-Cole transform, $\mathfrak{h}(t,r) = \log \mathcal{Z}(t,r)$ solves KPZ equation

$$\frac{\partial}{\partial t}\mathfrak{h}(t,r) = \frac{\partial^2}{\partial r^2}\mathfrak{h}(t,r) + \left(\frac{\partial}{\partial r}\mathfrak{h}(t,r)\right)^2 + \eta(t,r)$$

What we will prove:

When observed on a line $m + n = Cst.$:

- ✘ The population concentrates around the highest peak and spreads at distance $O(1)$ around it.
- ✘ The second high peak does not contribute significantly.
- ✘ In large time, the population density converges (in distribution), without any scaling.

Outline

Boundary Conditions

Assign distinct weight distributions on the boundaries and in the bulk:

$$U_{i,0} = Y_{i,0} \text{ and } V_{0,j} = Y_{0,j} \text{ for } i,j \in \mathbb{N} := \{1,2,\dots\}.$$

Model b.c.(θ): For $\theta \in (0,\mu)$, denote by b.c.(θ) the model with

$$\begin{cases} \{U_{i,0}, V_{0,j}, Y_{i,j} : i,j \in \mathbb{N}\} \text{ are independent with distributions} \\ U_{i,0}^{-1} \sim \text{Gamma}(\theta), \quad V_{0,j}^{-1} \sim \text{Gamma}(\mu-\theta), \quad Y_{i,j}^{-1} \sim \text{Gamma}(\mu). \end{cases}$$

Recall the point-to-point partition function

$$Z_{m,n} = \sum_{\mathbf{x}:\mathbf{0}\mapsto(m,n)} \prod_{t=1}^{m+n} Y_{x_t},$$

and define new weights on horizontal or vertical edges

$$U_{m,n} = \frac{Z_{m,n}}{Z_{m-1,n}} \text{ and } V_{m,n} = \frac{Z_{m,n}}{Z_{m,n-1}}.$$

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Facts (Seppäläinen 2012): Along any down-right path the variables U, V's are mutually independent with marginal distributions

$$U^{-1} \sim \text{Gamma}(\theta), \; V^{-1} \sim \text{Gamma}(\mu-\theta) \quad .$$

Burke property for pedestrians

$$
\begin{aligned}
U_{1,1}^{-1} &= \frac{Z_{0,1}}{Z_{1,1}} \\
&= \frac{V_{0,1}}{(V_{0,1}+U_{1,0})Y_{1,1}} \\
&= \frac{U_{1,0}^{-1}}{U_{1,0}^{-1}+V_{0,1}^{-1}} \times Y_{1,1}^{-1}, \\
V_{1,1}^{-1} &= \frac{V_{0,1}^{-1}}{U_{1,0}^{-1}+V_{0,1}^{-1}} \times Y_{1,1}^{-1}
\end{aligned}
$$

Recall standard prop.: If u, v are independent Gamma(θ, resp. $\mu-\theta$), then

$$
\left(\frac{u}{u+v}, \frac{v}{u+v}\right) \sim \text{ Beta}(\theta, \mu-\theta) \text{ independent of } u+v \sim \text{ Gamma}(\mu)
$$

And conversely. Property $\implies$ above Fact.

Random walk representation for ratios of partition functions

Considering the down-right path along the vertices $x : x \cdot (\mathbf{e_1} + \mathbf{e_2}) = n$, we deduce the representation

$$\frac{Z_{k,n-k}}{Z_{0,n}} = \exp(-\sum_{i=0}^{k} X_i^n).$$

with a collection $X_k^n = -\log(\frac{U_{k,n-k}}{V_{k-1,n-k+1}})$ of i.i.d.r.v.'s. The endpoint distribution is

$$Q_n^\omega\{x_n = (k, n-k)\} = \frac{Z_{k,n-k}}{\sum_{i=0}^n Z_{i,n-i}} = \frac{1}{\sum_{i=0}^n \exp(-(S_i^n - S_k^n))}$$

with $\boxed{S_k^n = \sum_{i=1}^{k} X_i^n}$ a random walk; It is centered iff $\theta = \mu/2$.

The favorite endpoint is

$$I_n^n = \arg\min\{S_k^n; 0 \le k \le n\}.$$

Outline

main result

For every n, consider the end-point distribution centered around favorite endpoint,

$$\hat{\xi}_k^{(n)} = Q_n^{\omega}\left\{x_n = (l_n^n + k, n - l_n^n - k)\right\}, \quad k \in \mathbb{Z}.$$

Thus, $\hat{\xi}^{(n)} = (\hat{\xi}_k^{(n)}; k \in \mathbb{Z})$ is a random element of $\mathcal{M}_1(\mathbb{Z})$.

Theorem

For the model b.c.(θ) with $\theta \in (0, \mu)$, we have convergence in law

$$\hat{\xi}^{(n)} \xrightarrow{\mathcal{L}} \xi \qquad \text{in the space } (\mathcal{M}_1, \|\cdot\|_{TV}),$$

where $\|\mu - \nu\|_{TV} = \sum_k |\mu(k) - \nu(k)|$ is the total variation distance.

Consequences of main result: answers to our questions

A few consequences:

☞ Mass of favourite endpoint converges

$$\mathcal{I}_n \xrightarrow{\mathcal{L}} \max\{(\xi(k) + \xi(k+1))/2; k \in \mathbb{Z}\} > 0.$$

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$$\mathcal{I}_n \xrightarrow{\mathcal{L}} \max\{(\xi(k)+\xi(k+1))/2; k\in\mathbb{Z}\} > 0.$$

- Tightness of the endpoint: Letting $\overrightarrow{l_n} = (l_n, n-l_n)$,

$$\lim_{K\to\infty} \limsup_{n\to\infty} Q_n^\omega[|x_n - \overrightarrow{l_n}| \geq K] = 0 \quad \text{in probability}$$

Cf. uniqueness of geodesics, Newman 1995, . . . , Georgiou, Rassoul-Agha, Seppalainen 2015

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- Scaling limit of endpoint: By Donsker's invariance principle, when $\theta = \mu/2$, the RW S_k can be approximated by a Brownian motion W:

$$\frac{l_n}{n} \xrightarrow{\mathcal{L}} \arg\min_{t\in[0,1]} W_t \ ,$$

the arcsine law. (And so does $\frac{x_n}{n}$ by previous point.)

- When $\theta > \mu/2$, the limit is 0. In fact, if $\theta > \mu/2$, l_n converges in law to a finite limit. And symmetrically if $\theta < \mu/2$.

Consequences of main result, continued

Large deviations of the polymer endpoint:

- For $\theta = \mu/2$,

$$Q_n^\omega\{x_n = ([ns], n - [ns])\} \simeq e^{-\sqrt{n}[W(s) - \min_{[0,1]} W]},$$

- ...whereas for $\theta > \mu/2$,

$$Q_n^\omega\{x_n = ([ns], n - [ns])\} \simeq e^{-ns|\Psi_0(\theta) - \Psi_0(\mu-\theta)|}.$$

Observe:

- Change of speed in the LDP from equilibrium to non-equilibrium.
- Rate function is random in the first case (it depends on the environment), and deterministic in the second one. (*Differs from Barraquand's talk*)

b.c. are crucial

Observe that these results disagree with KPZ theory, namely on where localization takes place; but we believe that "the trapping at the minimum of a RW" is essentially the correct mechanism.

The disagrement come from the boundary conditions.

sketch of proof of main result for $\theta = \mu/2$

Recall that

$$Q_n^\omega\{x_n = (k, n-k)\} = \frac{1}{\sum_{i=0}^n \exp(-(S_i^n - S_k^n))}$$

Since we are only interested in the law of $Q_n^\omega\{x_n = (k, n-k)\}$, we drop the superscript n in S_n^n, X_i^n, I_n^n, etc. . .

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• Splitting a random walk S at its minimum is a well studied: Bertoin 1991-94, Doney 1989-94, in the mean-zero case and the drifted case.

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• Splitting a random walk S at its minimum is a well studied: Bertoin 1991-94, Doney 1989-94, in the mean-zero case and the drifted case.

• For $\theta = \mu/2$ (otherwise, quite different and simpler.): The process converges to 2 independent pieces on $\mathbb{Z}^+$ and $\mathbb{Z}^-$, glued at 0,

- (i) the first one being the random walk conditioned to stay non negative forever, say $S^\uparrow$;
- (ii) the other one being the time-reverse of the random walk with jumps $-X$ conditioned to stay strictly positive forever, say $S^\downarrow$.

For fixed K,

$$(S_{I_n+k} - S_{I_n})_{1\le k\le K} \xrightarrow{\mathcal{L}} (S_k^\uparrow)_{1\le k\le K} \ ,$$

$$(S_{I_n+k} - S_{I_n})_{-1\ge k\ge -K} \xrightarrow{\mathcal{L}} (S_k^\downarrow)_{1\le k\le K} \ .$$

sketch of proof of main result for $\theta = \mu/2$

Since we condition by a null event, $(S_k^{\uparrow})$ and $(S_k^{\downarrow})$ are taboo processes and a correct definition is via Doob's *h*-transform.

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Now, to control the full (unbounded) sum, we use a consequence of a result by Ritter 1981:

$$\lim_{\delta \to 0} \mathbb{P}[S_{k+l_n} - S_{l_n} > \delta k^{1/2-\epsilon} \text{ for all } k \leq n - l_n] = 1 \ .$$

With the preceeding, we get:

$$\frac{1}{\sum_{k=0}^{n} \exp(-(S_k - S_{l_n}))} \xrightarrow{\mathcal{L}} \frac{1}{\sum_{k=0}^{\infty} \exp(-S_k^{\uparrow}) + \sum_{k=1}^{\infty} \exp(-S_k^{\downarrow})} ,$$

which writes also

$$Q_n^{\omega}\{x_n = (l_n^n, n - l_n^n)\} = \hat{\xi}_0^n \xrightarrow{\mathcal{L}} \xi_0 .$$

sketch of proof of main result for $\theta = \mu/2$

For general values of k, the same arguments lead to the expression of the limit of $\xi^n_{l_n+k} = \hat{\xi}^n_k$, explicitely:

$$\xi_k = \begin{cases} \dfrac{\exp(-S^{\uparrow}_k)}{1 + \sum_{i=1}^{\infty} \exp(-S^{\uparrow}_i) + \sum_{i=1}^{\infty} \exp(-S^{\downarrow}_i)}, & \text{if } k \geq 0 \\ \dfrac{\exp(-S^{\downarrow}_k)}{1 + \sum_{i=1}^{\infty} \exp(-S^{\uparrow}_i) + \sum_{i=1}^{\infty} \exp(-S^{\downarrow}_i)}, & \text{if } k < 0 \end{cases}$$

□

Outline

Point to point polymer

Fix $\mu > 0$, $(p,q) \in (\mathbb{Z}_+^*)^2$ and for $N \in \mathbb{N}$, let R_N be the rectangle with diagonal $(0,0),(pN,qN)$.

Model P2P-b.c.(θ): Assume

$$\begin{cases} Y_{i,j} : (i,j) \in R_N \setminus \{\mathbf{0},(pN,qN)\} \text{ are independent with} \\ Y_{i,j}^{-1} \sim \mathrm{Gamma}(\theta,1) \text{ for } j \in \{0,qN\}, \\ Y_{i,j}^{-1} \sim \mathrm{Gamma}(\mu-\theta,1) \text{ for } i \in \{0,pN\}, \\ Y_{i,j}^{-1} \sim \mathrm{Gamma}(\mu,1) \text{ for } 0 < i < pN \text{ and } 0 < j < qN. \end{cases}$$

Point-to-point polymer measure is the probability measure

$$Q^{\omega}_{pN,qN}(\mathbf{x}) = \frac{1}{Z^{\omega}_{pN,qN}} \exp\Big\{\sum_{t=1}^{n} \omega(x_t)\Big\}.$$

For a path $\mathbf{x}$ denote by t^- the "time it crosses the second diagonal". The transverse coordinate of the crossing point can be described by

$$F(\mathbf{x}) = (x_{t^-} + x_{t^-+1}) \cdot (q\mathbf{e_1} - p\mathbf{e_2}).$$

Point to point polymer

Theorem

For any $\theta \in (0, \mu)$, there exist a random integer m_N depending on ω and a random probability measure $\hat{\xi}$ on $\mathbb{Z}$ such that, as $N \to \infty$,

$$\left(Q^{\omega}_{pN,qN}(F(\mathbf{x}) = m_N + k); k \in \mathbb{Z} \right) \xrightarrow{\mathcal{L}} \hat{\xi},$$

in the space $(\mathcal{M}_1, \|\cdot\|_{TV})$.

- Recall that middle-point localization for the point-to-point measure is not covered by the usual semi-martingale approach to localization, and this result is totally new.

Time is out

Perspectives:

- Model without boundaries
- Path localization
- Other solvable models

Time is out

Perspectives:

- ✌ Model without boundaries
- ✌ Path localization
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♠

Merci de votre attention !