Semi-Geostrophic theory, the Navier-Stokes Equations, and Kähler Geometry

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Outline

• Semi-geostrophic theory – Monge-Ampère equations, Legendre duality and Hamiltonian structure
• Higher-order balanced models
• Complex structures and Monge-Ampère
• Kähler geometry and the 2d incompressible Navier-Stokes equations
• Complex structures and the 3d Navier-Stokes equations
Semi-geostrophic theory

- Jets and fronts – two length scales
Semi-geostrophic equations: shallow water

\[
\frac{D u_g}{D t} - f v + g \frac{\partial h}{\partial x} = 0,
\frac{D v_g}{D t} + f u + g \frac{\partial h}{\partial y} = 0
\]

\[
\frac{D h}{D t} + h \nabla \cdot \mathbf{u} = 0,
\]

\[
\frac{D}{D t} = \frac{\partial}{\partial t} + \mathbf{u} \cdot \nabla
\]

*h(x, y, t)* is the depth of the fluid

\[
\begin{align*}
 u_g &= -\frac{g}{f} \frac{\partial h}{\partial y}, \\
 v_g &= \frac{g}{f} \frac{\partial h}{\partial x}
\end{align*}
\]

Geostrophic wind
Conservation laws

- The SG equations conserve energy and potential vorticity, $q$

\[
q = \frac{1}{h} \left( f + \frac{\partial v_g}{\partial x} - \frac{\partial u_g}{\partial y} + \frac{1}{f} \frac{\partial (u_g, v_g)}{\partial (x, y)} \right)
\]

\[
\frac{Dq}{Dt} = 0
\]
Geostrophic momentum coordinates

\[ X \equiv (X, Y) \equiv \left[ x + \frac{v_g}{f}, y - \frac{u_g}{f} \right] \]

Equations of motion become

\[ \frac{DX}{Dt} = u_g \equiv (u_g, v_g) \]

Potential vorticity

\[ q = \frac{f \partial (X, Y)}{h \partial (x, y)} \]
Legendre transformation

Define

\[ \phi = \frac{g}{f^2} h(x, y, t) \]

then

\[ P = \frac{1}{2} f \left( x^2 + y^2 \right) + \phi \]

and

\[ X = \nabla_x P \]

\[ R(X) = x \cdot X - P(x) \]

Singularities/Fronts

PV and a Monge-Ampère equation

\[ q(x, y, t) = \frac{g}{f \phi} \frac{\partial (X, Y)}{\partial (x, y)} \]
Figure 12. Example in §16 of $P[x,z]$ for $q = 1$. 

Geometric model (Cullen et al, 1984)
Hamiltonian structure

Define

\[ q^{-1} \equiv \rho(X) = \frac{f \phi}{g} \frac{\partial(x, y)}{\partial(X, Y)} \]
\[ = \frac{f \phi}{g} \left(R_{XX} R_{YY} - R_{XY}^2\right) \]

Then

\[ R(X, Y, t) = \Phi - \frac{1}{2}(X^2 + Y^2) \]
\[ \phi = \Phi(X, Y, t) - \frac{1}{2}(\Phi_X^2 + \Phi_Y^2) \]

\[ \frac{\partial \rho}{\partial t} + \dot{X} \frac{\partial \rho}{\partial X} + \dot{Y} \frac{\partial \rho}{\partial Y} = 0 \]

\[ \dot{X} = f \frac{\partial \Phi}{\partial Y}, \quad \dot{Y} = -f \frac{\partial \Phi}{\partial X} \]
Higher-order balanced models

There exists a family (Salmon 1985) of balanced models that conserve a PV of the form

\[ q = \frac{1}{h} \left( f + \left( \frac{\partial v_g}{\partial x} - \frac{\partial u_g}{\partial y} \right) + \frac{(1 - c^2)}{f} \frac{\partial (u_g, v_g)}{\partial (x, y)} \right) \]

McIntyre and Roulstone (1996)

\[ q = \frac{g}{f \phi} \frac{\partial (\chi, \overline{\mathcal{Y}})}{\partial (x, y)} \]

\[ \chi = x + icQ + P, \quad \overline{\mathcal{Y}} = y - icP + Q \]

\[ (P = \phi_x, Q = \phi_y) \]
Complex structure

Introduce a symplectic structure

\[ (\{x, y; P, Q\}, T^*\mathbb{R}^2, \Omega) \quad \Omega = dx \wedge dP + dy \wedge dQ \]

and a two-form

\[ \omega = AdP \wedge dy + B(dx \wedge dP - dy \wedge dQ) + Cdx \wedge dQ + DdP \wedge dQ + E dx \wedge dy \]

On the graph of \( \varphi \)

\[ P = \phi_x, Q = \phi_y \quad \omega|_\varphi = 0 \]

Monge-Ampère eqn

\[ A\phi_{xx} + 2B\phi_{xy} + C\phi_{yy} + D(\phi_{xx}\phi_{yy} - \phi_{xy}^2) + E = 0 \]
Define the Pfaffian

\[ \omega \wedge \omega = \text{pf}(\omega)\Omega \wedge \Omega \]

\[ \text{pf}(\omega) = AC - B^2 - DE \]

\[ \text{pf}(\omega) > 0 \]

then \( \omega \) (M-A eqn) is \textit{elliptic}, and

\[ I_{\mu \nu} = \frac{1}{\sqrt{\text{pf}(\omega)}}\Omega_{\mu \sigma} \omega_{\sigma \nu} \]

\[ I_\omega = \frac{1}{\sqrt{\text{pf}(\omega)}} \begin{pmatrix} B & -A & 0 & -D \\ C & -B & D & 0 \\ 0 & E & B & C \\ -E & 0 & -A & -B \end{pmatrix} \]

is an \textit{almost-complex} structure \( I_\omega^2 = -\text{Id} \)

\[ T^*\mathbb{R}^2, \omega, I_\omega \]

is an almost-Kähler manifold

(Delahais & Roulstone, Proc. R. Soc. Lond. 2009)
Legendrian Structure

(Delahaiés & R. (2009), R. & Sewell (2012))

\[ X_3 = \sqrt{1 - f^{-1}\xi_0} x + \frac{1 + i\sqrt{pf(\omega)}}{\sqrt{1 - f^{-1}\xi_0}} p \]

\[ Y_3 = \sqrt{1 - f^{-1}\xi_0} y + \frac{1 + i\sqrt{pf(\omega)}}{\sqrt{1 - f^{-1}\xi_0}} q, \]

\[ \mathcal{P}(x, y) = \phi + \frac{1}{2} \frac{1 - f^{-1}\xi_0}{1 + i\sqrt{pf(\omega)}} (x^2 + y^2), \]

\[ \mathcal{R}(X_3, Y_3) = \frac{\sqrt{1 - f^{-1}\xi_0}}{1 + i\sqrt{pf(\omega)}} (xX_3 + yY_3) - \mathcal{P} \]

\[ \Phi(X_3, Y_3) = -\mathcal{R} + \frac{1}{2} \frac{1}{1 + i\sqrt{pf(\omega)}} (X_3^2 + Y_3^2). \]
Incompressible Navier-Stokes

\[
\frac{\partial u}{\partial t} + u \cdot \nabla u + \nabla p = \nu \nabla^2 u
\]

\[
\nabla \cdot u = 0
\]

Apply \( \text{div } v = 0 \)

\[
-\nabla^2 p = u_{i,j} u_{j,i}
\]

2d: Stream function

\[
u = k \times \nabla \psi
\]

\[
\nabla^2 p = -2(\psi_{xy}^2 - \psi_{xx} \psi_{yy})
\]
Complex structure

Poisson eqn

\[ \omega \equiv \nabla^2 p \, dx \wedge dy - 2 du \wedge dv \]

Components

\[ \omega_{\mu\nu} = \begin{pmatrix} 0 & \nabla^2 p & 0 & 0 \\ -\nabla^2 p & 0 & 0 & 0 \\ 0 & 0 & 0 & -2 \\ 0 & 0 & 2 & 0 \end{pmatrix} \]

Complex structure

\[ I_{\mu\nu} = \frac{1}{\sqrt{pf(\omega)}} \Omega_{\mu\sigma} \omega_{\sigma\nu} \]

\[ \nabla^2 p = 2\alpha^2 \]

\[ I_{\omega} = I_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & -\frac{1}{\alpha} \\ 0 & 0 & \frac{1}{\alpha} & 0 \\ 0 & -\alpha & 0 & 0 \\ \alpha & 0 & 0 & 0 \end{pmatrix} \]

\[ I_{\omega} = -I \text{ if } \nabla^2 p > 0 \]
Fixing a volume form in terms of the symplectic structure, we define a metric

$$g_\omega(X, Y) = \frac{\iota_X \Omega \wedge \iota_Y \omega + \iota_Y \Omega \wedge \iota_X \omega}{\Omega \wedge \Omega} \wedge \pi^*(\text{vol}), \quad X, Y \in T\mathbb{R}^n,$$

We construct a metrically-dual two-form for the hyperbolic MA equation.

Taking the Lie derivative of $\omega$ along the flow $u$

$$\mathcal{L}_u \omega = (P_{xx} - P_{yy})(du \wedge dy + dv \wedge dx) + 2P_{xy}(dx \wedge du - dy \wedge dv).$$
Vorticity and Rate of Strain (Weiss Criterion)

\[ Q = \frac{1}{2} (W_{ij} W_{ij} - S_{ij} S_{ij}) = \frac{1}{4} (\zeta^2 - 2S_{ij} S_{ij}) \]

\[ S_{ij} = \frac{1}{2} (u_{i,j} + u_{j,i}) \]

\[ W_{ij} = \frac{1}{2} (u_{i,j} - u_{j,i}) \]

\[ \zeta = \nabla \times \mathbf{u} \]

\[ Q > 0 \text{ implies almost-complex structure (ellipticity)} \]
A geometric interpretation of coherent structures in Navier–Stokes flows

By I. Roulstone¹,*, B. Banos², J. D. Gibbon³ and V. N. Roubtsov⁴,⁵
J.D. Gibbon (Physica D 2008 – Euler, 250 years on): “The elliptic equation for the pressure is by no means fully understood and locally holds the key to the formation of vortical structures through the sign of the Laplacian of pressure. In this relation, which is often thought of as a constraint, may lie a deeper knowledge of the geometry of both the Euler and Navier-Stokes equations...The fact that vortex structures are dynamically favoured may be explained by inherent geometrical properties of the Euler equations but little is known about these features.”

\[
\Delta p = -u_{i,j}u_{j,i} = \frac{1}{2}\zeta^2 - \text{Tr} \ S^2
\]
Geometry of 3-forms (Hitchin)

\[ \Delta p = -u_{i,j}u_{j,i} = \frac{1}{2}\zeta^2 - \text{Tr} S^2 \]

\[ \sigma = \Delta p dx_1 \wedge dx_2 \wedge dx_3 - 2(du_1 \wedge du_2 \wedge dx_3 + du_1 \wedge dx_2 \wedge du_3 + dx_1 \wedge du_2 \wedge du_3) \]

Lychagin-Rubtsov (LR) metric

\[ q_\sigma(w, w) \equiv -\frac{1}{4} \perp^2 (\iota(w)\sigma \wedge \iota(w)\sigma) \]

\[ \perp \sigma = \iota \left( \frac{\partial}{\partial x} \wedge \frac{\partial}{\partial u} \right) \sigma. \]
Metric and Pfaffian

Construct a linear operator, $K_\omega$, using LR metric and symplectic structure

The “pfaffian”
Complex structure

In particular, when $\lambda(\varpi)<0$, the tensor

$$J_{\varpi} = \frac{1}{\sqrt{-\lambda(\varpi)}} K_{\varpi}$$

is an almost-complex structure and the real three-form $\varpi$ is the real part of the complex form

$$\varpi^c = \varpi + i \hat{\omega},$$

$$\varpi = (\mu_1 + iv_1) \wedge (\mu_2 + iv_2) \wedge (\mu_3 + iv_3) + (\mu_1 - iv_1) \wedge (\mu_2 - iv_2) \wedge (\mu_3 - iv_3)$$

$$\equiv \alpha + \bar{\alpha}$$

where $\mu_i = (\Delta p/2)^{1/3} \, dx_i$ and $v_i = (\Delta p/2)^{-1/6} \, du_i$, when $\Delta p > 0$
Summary

- Vorticity-dominated incompressible Euler flows in 2D are associated with almost-Kähler structure – a geometric version of the “Weiss criterion”, much studied in turbulence.
- Using the geometry of 3-forms in six dimensions, we are able to generalize this criterion to 3D incompressible flows.
• These ideas originate in models are large-scale atmospheric flows, in which rotation dominates and an elliptic pde relates the flow velocity to the pressure field

• Roubtsov and R (1997, 2001), Delahaies and R (2009) showed how hyper-Kähler structures provide a geometric foundation for understanding Legendre duality (singularity theory), Hamiltonian structure and Monge-Ampère equations, in semi-geostrophic theory