Calibrated Submanifolds

Problem Sheet 2

1. (a) Let η be a calibration so that $\mathrm{d}^*\eta = 0$. Show that $*\eta$ is a calibration and describe the relationship between the calibrated planes of η and those of $*\eta$.

   (b) Show that the G_2 form φ and its dual $*\varphi$ are calibrations on $\mathbb{R}^7$.

   (c) Show that the Spin(7) form Φ is a calibration on $\mathbb{R}^8$.

2. Let $u, v : \mathbb{R}^2 \to \mathbb{R}$, let x, y be coordinates on $\mathbb{R}^2$ and consider $N = \mathrm{Graph}(u + iv) \subseteq \mathbb{C}^2$.

   (a) Show that if ω is the Kähler form on $\mathbb{C}^2$ and e_1, e_2 are orthogonal unit tangent vectors on N then
   $$|\omega(e_1, e_2)| = \frac{|1 + u_x v_y - u_y v_x|}{\sqrt{(1 + u_x^2 + v_x^2)(1 + u_y^2 + v_y^2)}}.$$

   (b) Use (a) to show that N is calibrated by ω if and only if u, v satisfy the Cauchy–Riemann equations.

3. Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be a smooth function and let $N = \mathrm{Graph}(f) \subseteq \mathbb{R}^{2n} = \mathbb{C}^n$.

   (a) Show that N is Lagrangian if and only if there exists a function $F : \mathbb{R}^n \to \mathbb{R}$ such that $f = \nabla F$. Show further that N is special Lagrangian if and only if $\mathrm{Im} \det_{\mathbb{C}}(I + i \,\mathrm{Hess}\, F) = 0$, where $\mathrm{Hess}\, F = (\frac{\partial^2 F}{\partial x_i \partial x_j})$.

   (b) If $n = 2$, show that N is special Lagrangian if and only if F is harmonic. Compare this result to Question 2.

   (c) If $n = 3$, show that N is special Lagrangian if and only if $\Delta F = \det \mathrm{Hess}\, F$.

4. Let $x_1, y_1, \ldots, x_n, y_n$ be coordinates on $\mathbb{R}^{2n}$. We call an n-form η on $\mathbb{R}^{2n}$ a torus form if η lies in the span of forms of type
   $$\mathrm{d}x_{i_1} \wedge \ldots \wedge \mathrm{d}x_{i_k} \wedge \mathrm{d}y_{j_1} \wedge \ldots \wedge \mathrm{d}y_{j_l}$$
   where $\{i_1, \ldots, i_k\} \cap \{j_1, \ldots, j_l\} = \emptyset$ and $\{i_1, \ldots, i_k\} \cup \{j_1, \ldots, j_l\} = \{1, \ldots, n\}$. Show, by induction on n, that a torus form is a calibration if and only if
   $$\eta(\cos\theta_1 e_1 + \sin\theta_1 e_{n+1}, \ldots, \cos\theta_n e_n + \sin\theta_n e_{2n}) \leq 1$$
   for all $\theta_1, \ldots, \theta_n \in \mathbb{R}$.