Hall-Littlewood positive homomorphisms of the algebra of symmetric functions

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Algebra of symmetric functions

$\{x_i\}_{i=1}^{\infty}$ — formal variables.
Newton power sums:

$$p_k := \sum_{i=1}^{\infty} x_i^k.$$

The algebra of symmetric functions $\Lambda := \mathbb{R}[p_1, p_2, \dots]$.
We shall consider homomorphisms (= multiplicative functionals) $\pi : \Lambda \to \mathbb{R}$.

Examples.

1) $x_1, x_2, \dots$ are nonnegative numbers such that $\sum_i x_i \leq 1$.
2) $\pi(p_1) = 1$, $\pi(p_k) = 0$ for $k \geq 2$.

Homomorphisms are in one to one correspondence with a sequence $\{\pi(p_k)\}_{k=1,2,\dots}$.

Hall-Littlewood and Schur functions

Let $\lambda = \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k$ be a partition (= Young diagram). For $t \in [0;1)$ a Hall-Littlewood function is determined by

$$Q_\lambda(x_1, \ldots, x_n; t) := c_{\lambda,t} \sum_{\sigma \in S_n} x_{\sigma(1)}^{\lambda_1} x_{\sigma(2)}^{\lambda_2} \ldots x_{\sigma(k)}^{\lambda_k} \prod_{i<j} \frac{x_{\sigma(i)} - t x_{\sigma(j)}}{x_{\sigma(i)} - x_{\sigma(j)}}.$$

For $t = 0$ Hall-Littlewood functions turn into Schur functions:

$$Q_\lambda(x_1, \ldots, x_n; 0) = s_\lambda(x_1, \ldots, x_n) := \frac{\det \left(x_i^{\lambda_j + k - j} \right)_{i,j=1}^n}{\prod_{1 \leq i < j \leq n} (x_i - x_j)}$$

One can define $Q_\lambda \in \Lambda$, $s_\lambda \in \Lambda$.

Hall-Littlewood positive homomorphisms

We shall say that $\pi : \Lambda \to \mathbb{R}$ is a *Hall-Littlewood positive* homomorphism if for any partition λ the value $\pi(Q_\lambda)$ is nonnegative.

We shall say that $\pi : \Lambda \to \mathbb{R}$ is a *Schur positive* homomorphism if for any partition λ the value $\pi(s_\lambda)$ is nonnegative.

HL-positive homomorphisms.

1) $x_1, x_2, \ldots$ are nonnegative numbers such that $\sum_i x_i \leq 1$.

2) $\pi(p_1) = 1$, $\pi(p_k) = 0$ for $k \geq 2$.

Classification

We are interested in the description of all Hall-Littlewood positive homomorphisms.
This question arises in different contexts:

- Schur case: representation theory of the infinite symmetric group.
- Schur case: totally positive Toeplitz matrices.
- Hall-Littlewood case: asymptotic representation theory of the general linear groups over a finite field.

Also Schur (or Macdonald) positive homomorphisms play an important role in the construction of Schur (or Macdonald) processes.

Schur case: Thoma theorem

Theorem (Thoma '64, Edrei'53, Vershik-Kerov'81)

All Schur-positive homomorphisms of Λ *are parameterized by* $\gamma \geq 0$, *and* $\alpha = (\alpha_1 \geq \alpha_2 \geq \ldots \geq 0)$, $\beta = (\beta_1 \geq \beta_2 \geq \ldots \geq 0)$, *such that* $\sum_{i=1}^{\infty}(\alpha_i + \beta_i) < \infty$, *and*

$$\pi_{\alpha,\beta}(p_1) = \sum_{i=1}^{\infty}(\alpha_i + \beta_i) + \gamma,$$

$$\pi_{\alpha,\beta}(p_k) = \sum_{i=1}^{\infty}\alpha_i^k + (-1)^{k-1}\sum_{i=1}^{\infty}\beta_i^k, \qquad k \geq 2$$

Kerov's conjecture

Conjecture (Kerov'93)

For $0 \leq t < 1$ *all Hall-Littlewood positive homomorhpisms are parameterized by* $\gamma \geq 0$, *and* $\alpha = (\alpha_1 \geq \alpha_2 \geq \ldots \geq 0)$, $\beta = (\beta_1 \geq \beta_2 \geq \ldots \geq 0)$, *such that* $\sum_{i=1}^{\infty}(\alpha_i + \beta_i) < \infty$, *and*

$$\pi_{\alpha,\beta}(p_1) = \sum_{i=1}^{\infty} \alpha_i + \left(\gamma + \sum_{i=1}^{\infty} \beta_i\right) \frac{1}{1-t},$$

$$\pi_{\alpha,\beta}(p_k) = \sum_{i=1}^{\infty} \alpha_i^k + \frac{(-1)^{k-1}}{1-t^k} \sum_{i=1}^{\infty} \beta_i^k, \qquad k \geq 2.$$

Probability measures

$\mathbb{Y}_n$ — the set of all Young diagrams with n boxes.
Let $w = \{\alpha_i, \beta_j, \gamma\}$ denote our set of parameters. Without loss of generalization we assume that $\sum_i (\alpha_i + \frac{\beta_i}{1-t}) + \frac{\gamma}{1-t} = 1$.
w_{Pl} corresponds to $\gamma = 1 - t$; $\alpha_i = 0$, $\beta_j = 0$.
Let π_w be a HL-positive homomorphism. Let us define a probability measure on $\mathbb{Y}_n$ by

$$HL_n^w(\lambda) := n! \pi_{w_{Pl}}(P_\lambda) \pi_w(Q_\lambda),$$

where P_λ is a "P"-Hall-Littlewood function (a constant multiple of Q_λ).
It turns out that the probabilistic properties of these measures are closely related to Kerov's conjecture.

Law of Large Numbers: Schur case

Schur case: $t = 0$.
Parameters $w = \{\alpha_i, \beta_j, \gamma\}$ have the following probabilistic meaning

Theorem (Vershik-Kerov'81)

Let $\lambda(n)$ be the random Young diagram with n boxes distributed according to HL_n^w. As $n \to \infty$, we have

$$\frac{\lambda_i(n)}{n} \to \alpha_i, \qquad \frac{\lambda'_j(n)}{n} \to \beta_j, \qquad \text{convergence in probability.}$$

where $\lambda_i(n)$ is the length of the i-th row of λ, and $\lambda'_j(n)$ is the length of the j-th column of λ.

LLN: Hall-Littlewood case

$w = \{\alpha_i, \beta_j, \gamma\}$. Let λ be a random Young diagram distributed according to HL_n^w, and let $\gamma = 0$.

Theorem (Bufetov-Petrov'14)

As $n \to \infty$, we have

$$\frac{\lambda_i(n)}{n} \to \alpha_i, \quad \frac{\lambda_j'(n)}{n} \to \frac{\beta_j}{1-t},$$

convergence in probability.

Method of proof: — new dynamics (generalization of RSK) which samples HL_n^w; it is constructed with the use of Borodin-Petrov'13 (similar dynamics were constructed in O'Connell-Pei'12; related to q-Whittaker functions).
— Analysis of this dynamics.

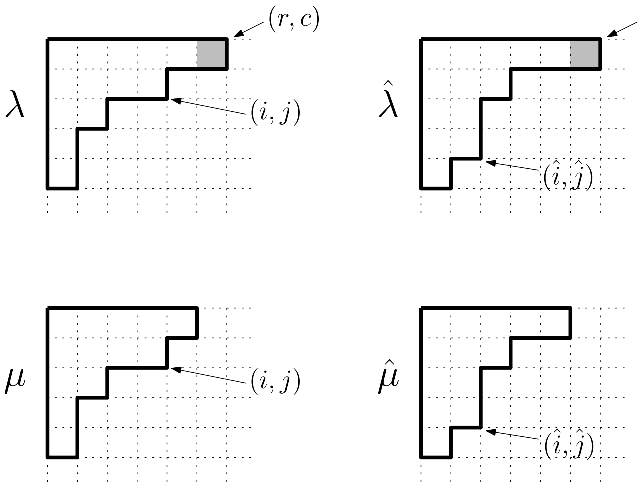

Figure: An example of $\lambda, \hat{\lambda}$ and $\mu, \hat{\mu}$. Here the gray box is $(r, c) = (1, 6)$, and $(i, j) = (2, 4)$, $(\hat{i}, \hat{j}) = (4, 2)$.

1^N stands for $\underbrace{(1,1,\ldots,1)}_{N}$. Assume that $N \gg 1$.

Conjecture (Bufetov-Gorin'14)

Let $\lambda, \hat{\lambda} \in \mathbb{Y}_n$ and $\mu, \hat{\mu} \in \mathbb{Y}_{n-1}$ be two pairs of Young diagrams such that both λ,$\hat{\lambda}$ and μ,$\hat{\mu}$ differ by the move of box (i,j) into the position $(\hat{i},\hat{j})$ with $\hat{i} > i$. Further, assume that $\lambda \setminus \mu = \hat{\lambda} \setminus \hat{\mu} = (r,c)$. If $r < i$, then

$$\left(1 - t^{\hat{\lambda}'_c - \hat{\lambda}'_{c+1}}\right) \frac{Q_{\hat{\mu}}(1^N;t)}{Q_{\hat{\lambda}}(1^N;t)} \geq \left(1 - t^{\lambda'_c - \lambda'_{c+1}}\right) \frac{Q_{\mu}(1^N;t)}{Q_{\lambda}(1^N;t)}.$$

Proposition (Bufetov-Gorin'14)

The conjecture above implies Kerov's conjecture.

Schur case

Proposition (Bufetov-Gorin'14)

Let $\lambda, \hat{\lambda} \in \mathbb{Y}_n$ and $\mu, \hat{\mu} \in \mathbb{Y}_{n-1}$ be two pairs of Young diagrams, such that both λ,$\hat{\lambda}$ and μ,$\hat{\mu}$ differ by the move of box (i, j) into the position $(\hat{i}, \hat{j})$ with $\hat{i} > i$. Further, assume that $\lambda \setminus \mu = \hat{\lambda} \setminus \hat{\mu} = (r, c)$. If $r < i$, then

$$s_\lambda(1^N)s_{\hat{\mu}}(1^N) \geq s_{\hat{\lambda}}(1^N)s_\mu(1^N).$$

This proposition and considerations of Bufetov-Gorin'14 give a new proof of Thoma theorem.

More information

More details can be found in

- A. Bufetov, L. Petrov, “Law of Large Numbers for Infinite Random Matrices over a Finite Field”, arXiv:1402.1772, to appear in Selecta Mathematica.
- A. Bufetov, V. Gorin, “Stochastic monotonicity in Young graph and Thoma theorem”, arXiv:1411.3307, to appear in IMRN.