Random walks in Beta random environment

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(Based on joint works with Ivan Corwin)

Consider the simple random walk X_t on $\mathbb{Z}$, starting from 0. We note

$$\mathbb{P}\big(X_{t+1} = X_t + 1\big) = \frac{\alpha}{\alpha+\beta}, \quad \mathbb{P}\big(X_{t+1} = X_t - 1\big) = \frac{\beta}{\alpha+\beta}.$$

The Central Limit Theorem says that

$$\frac{X_t - t\frac{\alpha-\beta}{\alpha+\beta}}{\sigma\sqrt{t}} \Longrightarrow \mathcal{N}(0,1).$$

Theorem (Cramér)

For $\frac{\alpha-\beta}{\alpha+\beta} < x < 1$*,*

$$\frac{\log\Big(\mathbb{P}(X_t > xt)\Big)}{t} \xrightarrow[t\to\infty]{} -I(x),$$

where $I(x)$ *is the Legendre transform of*

$$\lambda(z) := \log\Big(\mathbb{E}\big[e^{zX_1}\big]\Big) = \log\left(\frac{\alpha e^z + \beta e^{-z}}{\alpha+\beta}\right).$$

In random environment ?

Question

What can we say for a random walk in random environment ?

In this talk, we consider simple random walks on $\mathbb{Z}$ in space-time i.i.d. environment:

$$\mathsf{P}\big(X_{t+1} = x+1 | X_t = x\big) = B_{t,x}, \quad \mathsf{P}\big(X_{t+1} = x-1 | X_t = x\big) = 1 - B_{t,x},$$

where $(B_{t,x})_{t,x}$ are i.i.d.
We note P, E (resp. $\mathbb{P}, \mathbb{E}$) the measure and expectation with respect to the random walk (resp. the environment)

Answer

All results from the previous slide still hold, even conditionally on the environment, for almost every realization of the environment.

Quenched large deviation principle

Theorem (Rassoul-Agha, Seppäläinen and Yilmaz, 2013)

Assume that $\log(B_{t,x})$ *have a finite third moment. Then, the limiting moment generating function*

$$\lambda(z) := \lim_{t\to\infty} \frac{1}{t} \log\left(\mathsf{E}\left[e^{zX_t}\right]\right),$$

exists a.s., and

$$\frac{\log\left(\mathsf{P}(X_t > xt)\right)}{t} \xrightarrow[t\to\infty]{a.s.} -I(x).$$

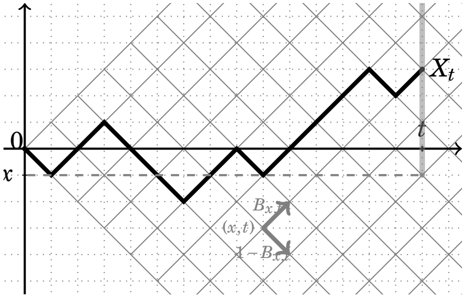
where $I(x)$ *is the Legendre transform of* λ*.*

An exactly solvable model: the Beta RWRE

We assume that $(B_{t,x})$ follow the $Beta(\alpha,\beta)$ distribution.

$$\mathbb{P}\Big(B \in [x, x + \mathrm{d}x]\Big) = x^{\alpha-1}(1-x)^{\beta-1}\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\mathrm{d}x.$$

- *Exactly solvable* means that we can exactly compute the law of $\mathsf{P}(X_t > xt)$ (and more).
- The annealed law of the Beta RWRE is the simple random walk from the first slide.



For simplicity, assume $\alpha = \beta = 1$. (Uniform case)

Theorem (B.-Corwin)

The LDP rate function is

$$I(x) = 1 - \sqrt{1 - x^2}.$$

We have the convergence in distribution as $t \to \infty$*,*

$$\frac{\log\left(\mathsf{P}\left(X_t > xt\right)\right) + I(x)t}{\sigma(x) \cdot t^{1/3}} \Longrightarrow \mathscr{L}_{GUE},$$

where $\mathscr{L}_{GUE}$ *is the GUE Tracy-Widom distribution, and*

$$\sigma(x)^3 = \frac{2I(x)^2}{1 - I(x)},$$

under the (technical) hypothesis that $x > 4/5$.

The theorem should extend to the general parameter case α, β and when x covers the full range of large deviation events (i.e. $x \in (0,1)$).

Fredholm determinant

Theorem (B.- Corwin)

Let $u \in \mathbb{C} \setminus \mathbb{R}_+$, and t, x with the same parity. Then for any parameters $\alpha, \beta > 0$ one has

$$\mathbb{E}\left[e^{u\mathsf{P}(X_t > x)}\right] = \det(I + K_u)_{\mathbb{L}^2(C_0)}$$

where C_0 is a small positively oriented circle containing 0 but not $-\alpha-\beta$ nor -1, and $K_u : \mathbb{L}^2(C_0) \to \mathbb{L}^2(C_0)$ is defined by its integral kernel

$$K_u(w, w') = \frac{1}{2i\pi} \int_{1/2-i\infty}^{1/2+i\infty} \frac{\pi}{\sin(\pi s)} (-u)^s \frac{g(w)}{g(w+s)} \frac{\mathrm{d}s}{s + w - w'}$$

where

$$g(w) = \left(\frac{\Gamma(w)}{\Gamma(\alpha + w)}\right)^{(t-x)/2} \left(\frac{\Gamma(\alpha + \beta + w)}{\Gamma(\alpha + w)}\right)^{(t+x)/2} \Gamma(w).$$

$$\det(I+K_u)_{\mathbb{L}^2(C_0)} := 1 + \sum_{n=1}^{\infty} \frac{1}{n!} \left(\frac{1}{2i\pi}\right)^n \int_{C_0} \dots \int_{C_0} \det\left[K_u(w_i, w_j)\right]_{i,j=1}^{n} \mathrm{d}w_1 \dots \mathrm{d}w_n.$$

Idea of the proof

Direct proof

1. Interpret the r.v. $\mathsf{P}(X_t > x)$ as the partition function $Z(t,x)$ of some polymer model (a particular random average process).
2. Find a recurrence relation for $Z(t,x)$.
3. It yields an evolution equation for $t \mapsto \mathbb{E}[Z(t,x_1)\dots Z(t,x_k)]$.
4. Solve the equation using a variant of Bethe ansatz.
5. Take moment generating series. It works !
6. Write it as a Fredholm determinant using ideas from Macdonald processes.

Origin

$Z(t,x)$ is a limit of observables of the q-Hahn TASEP, a Bethe ansatz solvable interacting particle system introduced by Povolotsky. (like the strict-weak polymer, cf Hao Shen's talk)

Extreme value theory

Fact

The order of the maximum of N i.i.d. random variables is the quantile or order $1-1/N$.

Relation LDP / extreme values

Second order corrections to the LDP have an interpretation in terms of second order fluctuations of the maximum of i.i.d. drawings.

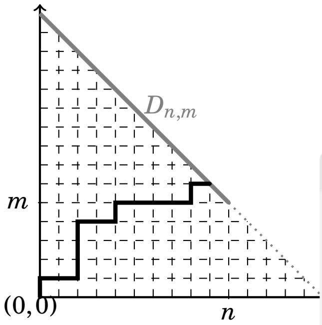
Corollary (B.-Corwin)

Let $X_t^{(1)},\dots,X_t^{(N)}$ be random walks drawn independently in the same environment. Set $N=e^{ct}$. Then, for $\alpha=\beta=1$,

$$\frac{\max_{i=1,\dots,e^{ct}}\left\{X_t^{(i)}\right\}-t\sqrt{1-(1-c)^2}}{d(c)\cdot t^{1/3}} \Longrightarrow \mathscr{L}_{GUE},$$

where $d(c)$ is an explicit function (proved under assumption $c>2/5$).

Zero temperature limit



We define the first passage-time $T(n,m)$ from $(0,0)$ to the half-line $D_{n,m}$ by

$$T(n,m) = \min_{\pi:(0,0)\to D_{n,m}} \sum_{e\in\pi} t_e$$

Passage times

For $(\xi_{i,j})$ i.i.d. Bernoulli and (E_e) i.i.d. Exponential,

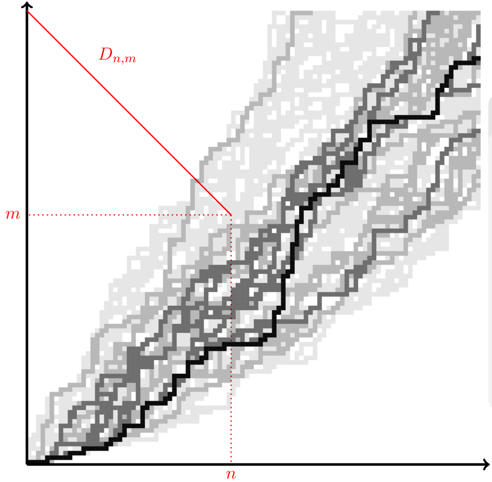
$$t_e = \begin{cases} \xi_{i,j} E_e & \text{if } e \text{ is horizontal,} \\ (1-\xi_{i,j}) E_e & \text{if } e \text{ is vertical.} \end{cases}$$

Theorem (B.-Corwin)

For any $\kappa > a/b$ and parameters $a, b > 0$, there exist constants $\rho(\kappa)$ and $\tau(\kappa)$, s.t.

$$\frac{T(n,\kappa n) - \tau(\kappa) n}{\rho(\kappa) n^{1/3}} \Longrightarrow \mathcal{L}_{GUE}.$$

Dynamical construction



Alternative description

- At time 0, only one random walk trajectory (in black).
- One adds to the percolation cluster portions of branching-coalescing random walks at exponential rate, at each branching point.

Outlook

We have seen

- A first exactly solvable model of space-time RWRE.
- Second order corrections to the LDP converge to $\mathscr{L}_{\mathrm{GUE}}$.
- Limit theorem for the max of $N = e^{ct}$ trajectories.
- Results propagate to the zero temperature model.

Questions

- KPZ universality for RWRE and random average process, to which extent ?
- Integrability : determinantal structure ? Analogue of Schur/Macdonald processes ? Link with a random matrix model ?
- Tracy-Widom distribution and extreme value theory...