Physics and G_2-manifolds

Bobby Samir Acharya

King's College London.
$\partial_\mu F^{\mu\nu} = j^\nu \;\; d\varphi = d * \varphi = 0$
and
ICTP Trieste $\bar{\Psi}(1-\gamma_5)\Psi$

The Rich Physics-Mathematics Interface

- **ALL** the known physics in our entire Universe is *extremely well* modelled by the Standard Model of Particle physics plus General Relativity
- This **mathematical** model is based on these equations:
  - Maxwell equations
  - Yang-Mills equations
  - Dirac Equation
  - Klein-Gordon Equation (Higgs equation)
  - Einstein Equations
- The impact of these equations upon mathematics, geometry in particular, cannot by emphasized enough
  - Index theory
  - Moduli spaces (Monopoles, Instantons, Higgs bundles, flat connections etc)
  - Donalsdon Theory, Seiberg-Witten
  - Einstein manifolds
  - This is just a selection of some 20th century highlights; many others

What is String Theory?

- String theory is based on equations which describe 2d surfaces minimally embedded in space-time
- A remarkable fact is that the equations which describe the low energy harmonics of such strings include:
  - Maxwell equations
  - Yang-Mills equations
  - Dirac Equation
  - Klein-Gordon Equation (Higgs equation)
  - Einstein Equations
- This is why String theory is considered our best candidate for a unified description of nature
- One of the main goals is to understand solutions of string theory which look identical to our Universe

Superstring theories

- The symmetries of string theory strongly restrict the number of consistent quantum string theories
- Also, from this list "superstring" theories are the best understood.
- There are just five superstring theories:
  - Type IIA
  - Type IIB
  - Type I
  - $E_8 \times E_8$ Heterotic
  - $Spin(32)/\mathbf{Z_2}$ Heterotic
- All are defined on ten dimensional spacetime i.e. Lorentzian 10-manifolds
- They differ from each other by gauge symmetries, particle spectrum etc
- They all have supersymmetry –implies relations amongst equations e.g. Yang-Mills $\leftrightarrow$ Dirac

Superstring theories

- Superstring theories are defined (at weak coupling and in smooth regions of space) in ten Lorentzian dimensions ($M^{9,1}$
- We have physically observed only **three** dimensions of space so the **six extra dimensions** must be hidden
- Simplest solution is: $M^{9,1} = Z^6 \times R^{3,1}$ with product metric, Z^6 compact and $R^{3,1}$ approx flat.
- Simplest solution requires $g(Z)$ to be *Ricci flat* i.e. $Ric(g(Z)) = 0$
- Then we have, assuming (Cheeger-Gromoll splitting) Z is simply connected, that
  - $Hol(g(Z)) = O(6)$, $SO(6)$, or $SU(3)$
  - Only *known* examples of this type have $Hol(g(Z)) = SU(3)$
  - Yau's solution of Calabi conjecture generates examples
  - Calabi-Yau manifolds are supersymmetric: admit parallel spinor, θ, $\nabla_{g(Z)}\theta = 0$
- Question: are there compact, simply connected, Ricci flat manifolds with generic holonomy?
- Less simple solutions: "generalised Calabi-Yau", solutions of "Strominger system" (Hitchin)

Calabi-Yau manifolds and Mirror symmetry

- In the mid 80's Candelas, de la Ossa, Green and Parke realised
- Type IIA on a CY Z^6 has identical physics to Type IIB on a different CY $\hat{Z}^6$
- Remarkably, "classical" physics on $\hat{Z}^6 =$ "quantum" physics on Z^6.
  - "Classical" physics here are the period integrals of holomorphic 3-form as functions of complex structure moduli
  - "Quantum" physics here are "instantons" i.e. holomorphic maps from Riemann surfaces to Z^6
  - Numbers of holomorphic curves (Gromov-Witten invariants) related to period integrals
- Later, Strominger-Yau-Zaslow picture of CY's as T^3-fibrations in a limit lead to generalisations such as
- Coherent Sheaves on $Z \leftrightarrow$ special Lagrangians in $\hat{Z}$ plus flat connection (Fukaya)

Superstring dualities and M theory

- In the early 90's was realised that the five superstring theories are inter-related (Sen, Hull-Townsend, Witten)
- A "new" theory emerged which is required in order to "interpolate" between the string theories
- This, so-called, M theory resides in eleven Lorentzian dimensions (so *seven extra dimensions*).
- The relevant fields of M theory on $M^{10,1}$ are a metric g, a 3-form C, and a gravitino Ψ (a vector with values in the spin bundle.)
  - E.g. M theory on $S^1 \times M^{9,1} \longrightarrow$ Type IIA on $M^{9,1}$ in small S^1 limit
  - E.g. M theory on $I \times M^{9,1} \longrightarrow E_8$ Heterotic on $M^{9,1}$ in small I limit (I is an interval)
- "Simplest" solutions of M theory with 7 compact dimensions have $g(M^{10,1}) = g(X^7) + g(R^{3,1})$ with X compact and Ricci flat
- Only *known* examples of this kind with $\pi_1(X) = 0$ have holonomy G_2. (Joyce, Kovalev, Corti-Haskins-Nordstrom-Pacini)

G_2-manifolds in M theory

- G_2-manifolds (X) are models for the extra dimensions in M theory
- Smooth G_2-manifolds are not so relevant for particle physics: no non-Abelian gauge symmetries and no chiral fermions – both key ingredients of the Standard Model of Particle physics
- Codimension four orbifold singularities (along associative Q) give rise to Yang-Mills fields
- Codimension seven conical singularities (on Q) give rise to quarks and leptons
- Associative submanifolds play a key role: gauge couplings, Yukawa couplings (associatives which intersect triples of codimension seven singularities)

Some physics references for G_2-manifolds

- A recent review on the particle physics/dark matter: BSA, G. Kane and P. Kumar arXiv:1204.2795
- The M theory interpretation of the Fermi data: BSA, G. Kane, P. Kumar, R. Lu, B, Zheng arXiv:1205.5789
- Codimension four singularities: BSA, arXiv/hep-th/9812205 and 0011089
- Codimension seven singularities and $U(1)$ quotients: M. Atiyah and E. Witten and BSA, E.Witten

G_2-manifolds and Particle Physics: singularities

- $\mathbf{R^4}/\Gamma_{ADE}$ has $3rk(ADE)$ moduli corresponding to the $rk(ADE)$ S^2's which desingularise the singularity preserving a Ricci flat metric.
- Pick one such S^2, fixing the remaining $3rk(ADE) - 3$ moduli 'at the origin'.
- This gives a 7d family of ALE spaces fibered over $\mathbf{R^3}$ with a 'more singular' fiber at the origin.
- Conjecture: this 7-manifold admits a G_2-holonomy metric
- For the A_1 case this is the Bryant-Salamon metric on $\mathbf{R}^+ \times \mathbf{CP^3}$
- For A_{N+1} with generic fiber A_N there should be a G_2-holonomy metric on $\mathbf{R}^+ \times \mathbf{WCP^3_{N,N,1,1}}$

More on the Local Models

- So there ought to be G_2-holonomy metrics on "ALE-fibrations over 3-manifolds"
- In the examples above there are also $U(1)$ fibrations:
- $\mathbf{R}^+ \times \mathbf{CP^3}$ admits a $U(1)$ action whose quotient is $\mathbf{R}^+ \times \mathbf{S^5} = \mathbf{R^6}$
- The fixed points are a pair of special Lagrangian 3-planes in $\mathbf{R^6}$ which meet at $SU(3)$ angles.
- $\mathbf{R}^+ \times \mathbf{WCP^3_{PPQQ}}$ admits a $U(1)$ action whose quotient is $\mathbf{R}^+ \times \mathbf{S^5} = \mathbf{R^6}$
- The fixed point set is the same, but now the '1st Chern classes' of the $U(1)$ are P and Q
- Locally modelled on $U(1) \rightarrow \mathbf{R^4} \rightarrow \mathbf{R^3}$.

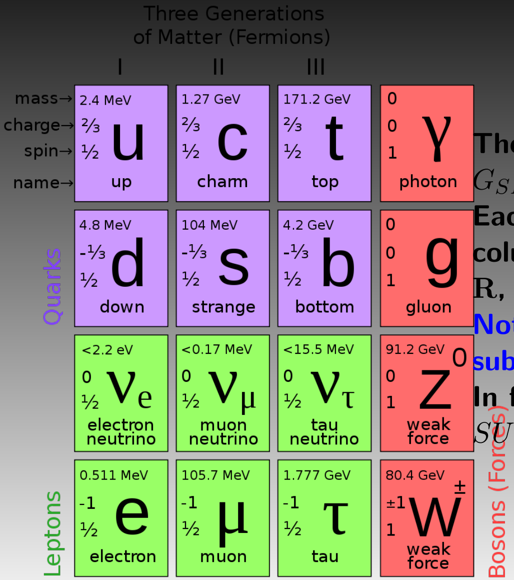
G_2-manifolds and Particle Physics: interactions

- The basic interactions in the Standard Model involve three particles
- Feynman diagrams have tri-valent vertices!
- There are interactions between the Higgs boson and pairs of fermions
- In M theory these correspond to an associative submanifold which passes through the three corresponding codim 7 singularities
- M theory analogues of "world-sheet instantons" aka hol. curves in a CY
- Is there a mathematical theory which counts such associative submanifolds?

Compact G_2 manifolds

- The bigger picture leads to compact G_2-manifolds with codim 4 and codim 7 singluarities
- K3-fibrations over S^3 emerge as natural candidates
- The generic fiber would have e.g. an $A_4 \times A_6 \times A_7$ ADE singularity
- Then at isolated points on the base the A_4 enhances to A_5 and D_5.
- Associative submanifolds which intersect the codim 7 singularities arise from the "network" of 2-spheres in the fibers as they vary over the S^3

The Standard Model of Particle Physics



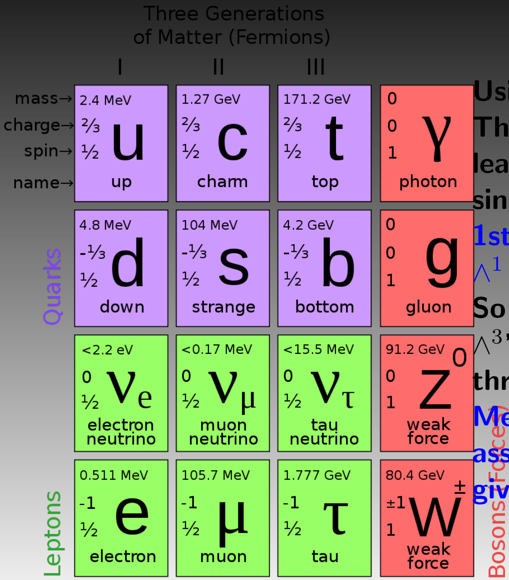
The gauge symmetry is $G_{SM} = SU(3)\times SU(2)\times U(1)$

Each family i.e. 1st three columns is in a 15-dim rep, R**, of** G_{SM}.

Note: G_{SM} **is a maximal subgroup of** $SU(5)$

In fact $\mathrm{R} \equiv \wedge^1 \oplus \wedge^3$ **in** $SU(5)$

The Standard Model of Particle Physics



Using $SU(5)$ language: The generic fiber has at least an $SU(5) = A_4$ singularity.

1st three columns are each $\wedge^1 \oplus \wedge^3$ of $SU(5)$

So three $\wedge^1$'s and three $\wedge^3$'s = 3 A_5 fibers and three D_5's.

Membrane instantons = associative submanifolds give hierarchy in masses

Other Issues which overlap with physics

- Would like to understand (much!) better the moduli space metric
- n.b. Kahler potential $K = -3ln(Vol(X))$, for complexified G_2-form: $\int_{H_3(X)} (\varphi + iC_3)$
- C_3 is the M theory 3-form 'potential'
- What are the geometric properties of the moduli space?
- **Important**: Can $Vol(X)$ be computed in practice as a function of the G_2 moduli space?
- Does the idea of an "ample" associative submanifold make sense, in analogy with "ample divisors"

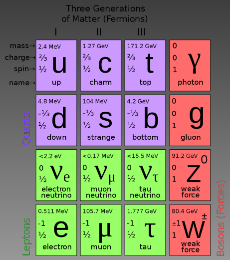
G_2-manifolds in string theory

- In string theory, G_2-manifolds also play a role
- D-branes in string theory are volume minimising submanifolds with gauge bundles on them
- D-branes wrapped on associative and co-associative submanifolds as well as along X itself are described by "topological field theory" equations:
- e.g. Casson invariants, Donaldson invariants, G_2-instantons, G_2-monopoles
- Key point: supersymmetric configurations (e.g. calibrations) $\leftrightarrow$ interesting equations

G_2-manifolds and the Real world I

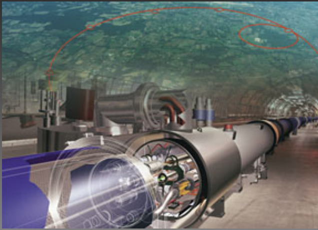
- Since they are odd-dimensional, G_2-manifolds are necessarily *real* and not complex!
- There is no Calabi-Yau theorem which allows algebraic geometry and complex analysis techniques to solve these problems
- One requires a strong emphasis on real differential geometry and analysis to make progress
- I would like to heartily encourage graduate students to stay focused on real analysis in this context!!
- Even though these sorts of problems are difficult, I have seen the field slowly changing over the past decade as young mathematicians have persisted to work on them as they become more senior!
- Perserverance will eventually reap rewards

G_2-manifolds and the LHC data



All known elementary particles

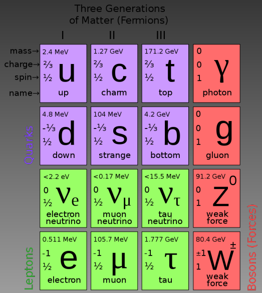
G_2-holonomy manifolds



Large Hadron Collider – smash protons into each other to examine the structure of matter.

Evidence for Higgs boson and G_2-manifolds

G_2-manifolds and the LHC data



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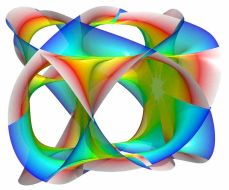
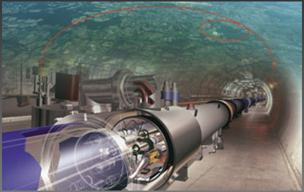


G_2-holonomy manifolds



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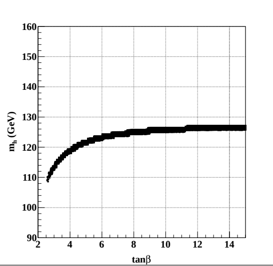
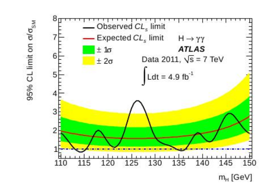
G_2-manifolds and the Real World II



Prediction for m_h from the physics of M theory on a G_2-manifold. Plot made summer 2011.



Data from the CERN LHC in Autumn 2011

Possible evidence for G_2 manifolds in LHC data?

Conclusions

- Life at the interface of theoretical physics and geometry is extremely interesting
- There are many interesting and significant problems in the field of G_2-manifolds and related areas
- Solving some of these would have a significant impact on physics

THANK YOU!