Recent Developments on Elliptic Curves
September 26 – 30, 2016

Abstracts of Talks

Mirela Ciperiani (University of Texas, Austin)

Tittle: Divisibility questions for genus one curves

Abstract: Genus one curves with a fixed Jacobian can be viewed as elements of the Weil-Chatelet group. We will discuss divisibility questions within this group. This leads us to analyzing the divisibility properties of the Tate-Shafarevich group. There are two related questions: 1. Are the elements of the Tate-Shafarevich group divisible within the Weil-Chatelet group? (Cassels' question.) 2. How does the Tate-Shafarevich group intersect the maximal divisible subgroup of the Weil-Chatelet group? (Bashmakov's question.) We will discuss our answers to these questions. This is joint work with Jakob Stix.

Henri Darmon (McGill University)

Title: Special elements and the Birch and Swinnerton-Dyer conjecture

Abstract: The notion of *special elements* and their relation with *L*-series has led to important progress on the Birch and Swinnerton-Dyer conjecture over the years, notably

a) The work of Coates and Wiles on the arithmetic of elliptic curves with complex multiplication via the connection between *elliptic units* and special values of the associated Katz *p*-adic *L*-function;

b) the proof by Gross-Zagier and Kolyvagin of the weak Birch and Swinnerton-Dyer conjecture for (modular) elliptic curves of analytic rank ≤ 1, which rests on *Heegner points* and their connections with Rankin *L*-series attached to *E* and an auxiliary quadratic imaginary field;

c) the proof by Kato of the weak Birch and Swinnerton-Dyer conjecture for modular elliptic curves twisted by arbitrary Dirichlet characters in analytic rank *0*, which exploits the connection between Beilinson elements in the second *K*-groups of modular curves and the Mazur-Swinnerton-Dyer *p*-adic *L*-function of an elliptic curve.

I will compare and contrast these three approaches, describe how the special elements of Beilinson-Kato fit into a larger trilogy comprising Beilinson-Flach elements and Gross-Kudla-Schoen diagonal cycles in *p*-adic families, and explain how these objects have led to some more recent progress on the Birch and Swinnerton-Dyer conjecture in analytic rank zero, for elliptic curves over $\mathbf{Q}$ viewed over the fields cut out by certain Artin representations of dimensions *2*, *3* and *4*.

Ellen Eischen (University of Oregon)

Title: p-adic L-functions and Eisenstein series

Abstract: This talk will give an overview of p-adic L-functions, focusing on p-adic L-functions for modular forms and Hecke characters. As part of the talk, I will give an overview of a construction of p-adic L-functions, which relies on relating them to p-adic families of Eisenstein series. These p-adic L-functions are one of the key ingredients in the main conjectures that will be discussed in Chris Skinner's and Xin Wan's talks. While most of the talk will be the in the setting of modular forms and elliptic curves, I will also briefly mention some recent developments in other settings and how they relate to these earlier constructions. This talk will be aimed at roughly the level of graduate students in number theory (but not experts on p-adic L-functions or Iwasawa theory).

Dick Gross (Harvard University)

Title: Heegner points

Abstract: After a brief introduction to the theory of complex multiplication, I will define Heegner points on the modular curve X_0(N). Bryan Birch initiated the study of the classes of divisors supported on the Heegner points and cusps in the Jacobian, and I will sketch the role these classes play in the descent at Eisenstein primes and the relation of their heights to the first derivative of Rankin L-series. I will describe some applications to the arithmetic of elliptic curves, and end with a discussion of some generalizations.

Wei Ho (University of Michigan)

Title: Selmer averages for families of elliptic curves

Abstract: We describe how to generalize the methods from Bhargava's and Shankar's talks to determine the average sizes of Selmer groups (and thus upper bounds for average ranks) in certain families of elliptic curves, e.g., those with various types of marked points. We discuss the general strategy as well as give many examples, including cases involving curves with marked points defined over extension fields.

Antonio Lei (Université Laval)

Title: Beilinson-Flach elements and finiteness of sha

Abstract: In this talk, I will briefly explain how the Beilinson-Flach elements introduced in Darmon's and Rotger's talks can be packaged to give Euler systems for Rankin-Selberg convolution of two modular forms over cyclotomic extensions of the field of rationals and for one modular form over ray class fields over an imaginary quadratic field. I will then discuss how to use these Euler systems to bound Selmer groups of elliptic curves in these settings. In particular, I will discuss how to show that these Selmer groups and sha are finite in certain cases.

Chao Li (Columbia University)

Title: Congruences between Heegner points and Goldfeld's conjecture

Abstract: We prove the weak Goldfeld conjecture for any elliptic curve E with a rational 3-isogeny, i.e., there is a positive proportion of quadratic twists of E with analytic rank 0 (resp. 1). We also prove the weak Goldfeld conjecture for the sextic twists family of j-invariant 0 curves. For a more general elliptic curve E, we show that the number of twists of E up to twisting discriminant X of analytic rank 0 (resp. 1) is >> X/log^{5/6}X, improving the current best general bound towards Goldfeld's conjecture due to Ono--Skinner (resp. Perelli--Pomykala). We prove these results by establishing a congruence formula between p-adic logarithms of Heegner points. This is joint work with Daniel Kriz.

Kartik Prasanna (University of Michigan)

Title: Cycles for Rankin-Selberg products and p-adic L-functions

Abstract: While the Birch–Swinnerton-Dyer conjecture concerns rational points on elliptic curves, recent progress on it has involved studying cycles on higher dimensional varieties. I will discuss this in a number of examples, explaining the connection with p-adic L-functions.

Victor Rotger (Universitat Politècnica de Catalunya)

Title: Beilinson-Kato, Beilinson-Flach and diagonal cycles: applications to BSD

Abstract: The aim of this lecture is to describe the theory of special elements arising from Beilinson-Kato classes, Beilinson-Flach classes and diagonal cycles in higher Chow groups of modular varieties. The global Galois cohomology classes associated to these elements can be made to vary along p-adic Hida families, yielding applications to the Birch and Swinnerton-Dyer conjecture in ranks 0, 1 and 2.

Arul Shankar (Harvard University)

Title: The average rank of elliptic curves is less than 1

Abstract: I will describe joint work with Manjul Bhargava in which we study the 5-Selmer groups in the family of all elliptic curves over Q. We will see how geometry-of-numbers methods can be used to show that the average size of the 5-Selmer groups of elliptic curves is 6. Using a result of Dokchitser--Dokchitser relating the root number of an elliptic curve to the parity of its 5-Selmer rank, we will then construct families of elliptic curves where the parity of the 5-Selmer rank is equidistributed. We will see how these two results can be combined to yield a bound of .885 on the average rank of elliptic curves.

Christopher Skinner (Princeton University)

Title: Iwasawa theory and the arithmetic of elliptic curves

Abstract: Iwasawa theory -- the study of the p-adic variation of Selmer groups and their related L-functions -- has featured prominently in progress on the Birch--Swinnerton-Dyer conjecture, beginning with the work of Coates and Wiles and continuing in the works of Rubin and Kato and even in recent works on the BSD formulas in ranks zero and one and on a converse to the results of Gross--Zager--Kolyvagin. In this talk we will recall some of the main conjectures for the Iwasawa theory of elliptic curves, the strategies of the proofs of some of these, and how these results have been combined with developments on Heegner points and the Gross--Zagier formula to provide criteria for elliptic curves to have both analytic rank and algebraic rank equal to zero or one.

Ye Tian (Chinese Academy of Sciences)

Title: Heegner points and congruent numbers

Abstract: It is conjectured that all positive integers n congruent to 5, 6, or 7 modulo 8 are congruent numbers and that among those integers there is a density one subset of n such that the elliptic curves ny^2=x^3-x have Mordell-Weil rank one. In this talk, we explain how one can show a positive density result using (general) Heegner points and their explicit Gross-Zagier and Waldspurger formulae. The talk is based on our joint work with Xinyi Yuan and Shouwu Zhang and work of Heath-Brown and Smith.

Eric Urban (Columbia University)

Title: On the rank of Selmer groups over **Q** of rational elliptic curves

Abstract: In this talk, I will discuss how we can bound below the rank of Selmer groups over **Q** of rational elliptic curves when their analytic rank may be greater than 1. This is an ongoing joint work with Chris Skinner.

Rodolfo Venerucci (Universität Duisburg-Essen)

Title: The anticyclotomic main conjectures for elliptic curves

Abstract: Let E be an elliptic curve defined over $\mathbf{Q}$, let $K/\mathbf{Q}$ be an imaginary quadratic field, and let p be a prime. The field K has a unique $\mathbf{Z}_p$-extension K_∞/K which is Galois and pro-dihedral over $\mathbf{Q}$, called the anticyclotomic $\mathbf{Z}_p$-extension. For every $n \in \mathbf{N}$, denote by K_n/K the cyclic sub-extension of K_∞/K of degree p^n.

To every character $\chi : \mathrm{Gal}(K_n/K) \to$ $\overline{\mathbf{Q}}_{p}^{\ast}$ is associated the complex L-function $L(E/K, \chi, s)$ of E/K twisted by χ. Roughly speaking, the anticyclotomic Iwasawa main conjectures — formulated in different settings by Perrin-Riou and by Bertolini--Darmon — relate the special values of $L(E/K, \chi, s)$ at $s = 1$, for varying χ, to the arithmetic of E over K_∞. More precisely, depending on the sign in the functional equation satisfied by $L(E/K, \chi, s)$, the algebraic part of $L(E/K, \chi, 1)$ or of $L'(E/K, \chi, 1)$ is related to the structure of the χ-part of the Selmer group of E/K_n. In this talk I will report on a joint work with Massimo Bertolini and Matteo Longo, in which we prove the anticyclotomic main conjectures for elliptic curves at primes of good reduction.

Xin Wan (Columbia University)

Title: Eisenstein congruences and Iwasawa main conjecture

Abstract: I'll give an overview of the key steps in proving the Rankin-Selberg Iwasawa main conjecture, using the congruences between Eisenstein series and cusp forms on rank 4 unitary groups.

Jerry Wang (Princeton University)

Title: A positive proportion of hyperelliptic curves over Q have no point over any odd degree extension

Abstract: In recent joint works with Manjul Bhargava and Benedict Gross, we showed that a positive proportion of hyperelliptic curves over Q of genus g have no points over any odd degree extension of Q. This is done by computing certain 2-Selmer averages and applying a result of Dokchitser-Dokchitser on the parity of the rank of the 2-Selmer groups in biquadratic twists. In this talk, we will see how arithmetic invariant theory and the geometric theory of pencils of quadrics are used to obtain the 2-Selmer averages.

Shou-wu Zhang (Princeton University)

Title: Gross—Zagier formula: extensions and applications

Abstract: In this talk, I will describe some extensions and applications of the Gross—Zagier formula and their proofs:
— A Gross—Zagier formula in full generality for arbitrary Shimura curves by Yuan—Zhang—Zhang;
— A Dirichlet class number formula for derivatives (or the Chowla—Selberg formula) by Xinyi Yuan with application to the André—Oort conjecture by Tsimerman;
— A Gross—Zagier formula over function fields for arbitrary derivatives by Zhiwei Yun and Wei Zhang including a Dirichlet class number formula for arbitrary derivatives.