Euler equations for rotating, Bressvaer stratified fluid (directly applicable to ocean - or atmosphere in pressure coordinates):

Given length scale $L$, velocity scale $U$, vertical scale $H$,
scale $e = U/L$ (Rossby scale), $e^2 = (H/L)^2$ (aspect ratio).
Then if $v^e$ is velocity, $p^e$ pressure, $\theta^e$ density,
and $\nabla = (\partial_1, \partial_2, e^2 \partial_3)$,
$v^e = (v_1, v_2, e^2 v_3)$
$\epsilon = (\partial_1 v^e + (v^e \cdot \nabla) v^e)_{12} + \nabla_{12} p^e + J v^e = 0$

$e^2 (\partial_3 v^e + (v^e \cdot \nabla) v^e)_{3} + \partial_3 p^e - \theta^e = 0$

$\nabla \cdot v^e = 0$

$\partial_3 \theta^e + (v^e \cdot \nabla) \theta^e = 0$

where $J = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

Solve with $v_0, \theta_0$ will $\epsilon \nabla \cdot \nabla \theta^e$. Consider family of solutions with varying $\epsilon$. 


Define the geostrophic velocity by
\[ \mathbf{v}_g = (-d_2 \rho, d_1 \rho, 0) \]

Then
\[ (\mathbf{v}_e, \mathbf{v}_e \cdot \mathbf{n}) = \mathbf{v}_g + e \left( \frac{d_1}{e_s} \mathbf{v}_e + (\mathbf{v}_e \cdot \nabla) \mathbf{v}_e \right) \]

so
\[ \mathbf{v}_e = \mathbf{v}_g + \mathbf{v}_e \frac{\partial}{\partial x} \frac{\partial}{\partial y} \]

Then substitute into Euler to get
\[ e \left( \frac{d_1}{e_s} \mathbf{v}_g + (\mathbf{v}_e \cdot \nabla) \mathbf{v}_g \right) + \nabla \mathbf{v}_e \cdot \mathbf{n} - \mathbf{v}_e \cdot \nabla \mathbf{v}_e = \nabla \cdot \mathbf{v}_e = 0 \]

\[ \frac{d_1}{e_s} \mathbf{v}_e + (\mathbf{v}_e \cdot \nabla) \mathbf{v}_e = 0 \]

\[ \frac{d_1}{e_s} \mathbf{v}_e + (\mathbf{v}_e \cdot \nabla) \mathbf{v}_e = 0 \]

Dropping \( o(\epsilon^3) \) terms gives the full equations

\[ \left( \mathbf{v}_e, \mathbf{\Theta} \right) = \left( -d_2 \rho, d_1 \rho, \frac{d_1}{e_s} \rho \right) \]

\[ e \left( \frac{d_1}{e_s} \mathbf{v}_g + (\mathbf{v}_e \cdot \nabla) \mathbf{v}_g \right)_{12} + \nabla \mathbf{v}_e + \mathbf{J} \mathbf{v}_e = 0 \]

\[ \nabla \cdot \mathbf{v}_e = 0 \]

\[ \frac{d_1}{e_s} \mathbf{v}_e + (\mathbf{v}_e \cdot \nabla) \mathbf{v}_e = 0 \]

Can write formally as

\[ e \left( \frac{d_1}{e_s} \mathbf{v}_g + (\mathbf{v}_e \cdot \nabla) \mathbf{v}_g \right) \mathbf{v}_e = \left( \begin{array}{c} -d_2 \rho \\ d_1 \rho \\ 0 \end{array} \right) \]
Define new variables
\[ P = \mathcal{E}_p + \frac{1}{2} (x_1^2 + x_2^2) \]
\[ X = \text{Lagrangian map} \]
\[ Y = X_{12} + (e \nu_2 \circ X, -e \nu_1 \circ X, e \Theta_0 X) \]
\[ = X_{12} + e \Delta P \circ X \]

SG becomes
\[ \Delta P \circ X = Y \]
\[ X \# L = L \text{ where } L \text{ is Lebesgue measure on } \Omega \]
\[ e \mathcal{E}_Y = J(Y - X) \] (\#)

(\#) can be solved if \( X \) known in terms of \( Y \).
Conserved energy of Euler is
\[ \int_{\Omega} \left( \frac{1}{2} \varepsilon (v \varepsilon)_{2}^{2} + \frac{1}{2} \varepsilon^{2} (v \varepsilon)_{3}^{2} - \Theta x_{3} \right) \, dx. \]

If \( S \varepsilon \) is
\[ \int_{\Omega} \frac{1}{2} \varepsilon (y \varepsilon)^{2} - \Theta x_{3} \, dx. \]

In Lagrangian variables, this is
\[ \varepsilon^{-1} \int_{\Omega} \left( x + y \right)_{2}^{2} - x \varepsilon y_{3} \, dx_{0} \]

\[ = \varepsilon^{-1} \int_{\Omega} C(x, y) \, dx_{0}. \]

Now reinterpret (*) as defining a flow map in \( Y \) space.

Given initial mass density \( \sigma_{0} \) in \( Y \) space defined by
\[ (y_{0})_{2} = \frac{1}{1} d + \varepsilon (v_{0}, -v), \]
\[ (y_{0})_{3} = c \varepsilon \theta_{0}, \]
\[ \sigma_{0} = y_{0} \# L. \]

Need to find \( X(t, \cdot) \) as a function of \( Y(t, \cdot) \), write as \( S \varepsilon Y \)
Mass density \( \sigma_{t} \) in \( Y \) space at time \( t \) is \( Y(t, \cdot) \# L. \)
If we require \( S \varepsilon \# \sigma(t, \cdot) = L \), this is consistent with \( X(t, \cdot) \# L = L \).
Choose $S_e$ s.t. the cost

$$\int c(x,y) \sigma_e \, d\Omega$$

is minimised under choices $f, S_e$ s.t. $S_e \neq \sigma_e = \Omega$.

Seek to show that this implies the remaining equation

$$Y = X_{12} + e \nabla p_0 X \quad \text{or} \quad \nabla p_0 X = Y$$

for some $p, p$.

This is an optimised transport problem. Solved by using the Kantorovich relaxation. Thus seek potentials $f(x), \phi(y)$ whereas $x \in X$ is Edain complete on $\Omega$ and $y \in Y$ is Edain complete on $Y$ space $\Omega$.

$$f(x) + \phi(y) \leq C(x,y)$$

and

$$\mathcal{K} = \int_{\Omega} f \, dx + \int_{\Omega} \phi \, dy \quad \text{is minimised}$$

The solution exists under reasonable assumptions and satisfies

$$f(x) + \phi(y) = C(x,y) \quad \forall x \in s(y)$$

In general

$$f(x) = \inf_y (c(x,y) - \phi(y))$$

$$\phi(y) = \inf_x (c(x,y) - f(x))$$

In our case,

$$C(x,y) = \frac{1}{2} (x - y)^2_{1,2} - x_3 y_3$$
In addition, we have
\[
\nabla \phi(x) = \frac{\partial}{\partial x} c(x,y) = (x-y)_2 - y_2
\]
\[
\nabla \psi(y) = \frac{\partial}{\partial y} c(x,y) = (y-x)_2 - x_2
\]
Define \( \epsilon \rho = -\phi \). Then
\[
\epsilon \rho \circ \gamma = \gamma - \gamma_{12} \text{ is required}
\]
and \( \nabla \rho \circ \gamma = \gamma \).

Also, have
\[
\phi(x) - \frac{1}{2} x_{12} = \inf \left(-x_2, y + \frac{1}{2}(y_{12} - y_{13})\right)
\]
so \( \phi(x) - \frac{1}{2} x_{12} = -\rho \) is concave as an infimum over planes. So \( \rho \) is convex and \( \nabla \rho \) is BV as is \( \nabla \rho \).

Similarly, \( R = \frac{1}{2} y_{12}^2 - \psi(y) \) is convex and \( \nabla R \) is BV. Define \( \epsilon = \epsilon \rho \) consistently with the definition of \( \rho \).

Now consider the evolution equation
\[
e \partial_t \gamma = (x_2 - \frac{1}{2}, y - x_1, 0) \in E \nabla \rho \circ \gamma
\]

We can write this as
\[
\partial_t \gamma + \nabla \phi(\gamma - U) = 0
\]
where \( U = \frac{1}{2} \nabla \rho \circ \gamma \) is BV and divergence free.

The aggregation principle shows that \( \gamma(t) \) can be solved.

The calculation of \( X \) for \( Y \) is equivalent to
the solution of a large system equation for \( R \), say
\[
X = \nabla R \circ \gamma
\]