Some Remarks on SLE and an Extended Sullivan Dictionary

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Today's lecture

A view of three fields from an analyst's (perhaps distorted) perspective.

Special spotlight on today's ceremony and the solution of the Ahlfors conjecture:

Ian Agol, Daniel Caligari, David Gabai

How to interpret tameness for other problems? (We will look at estimates for distortion.)

Acknowledgements

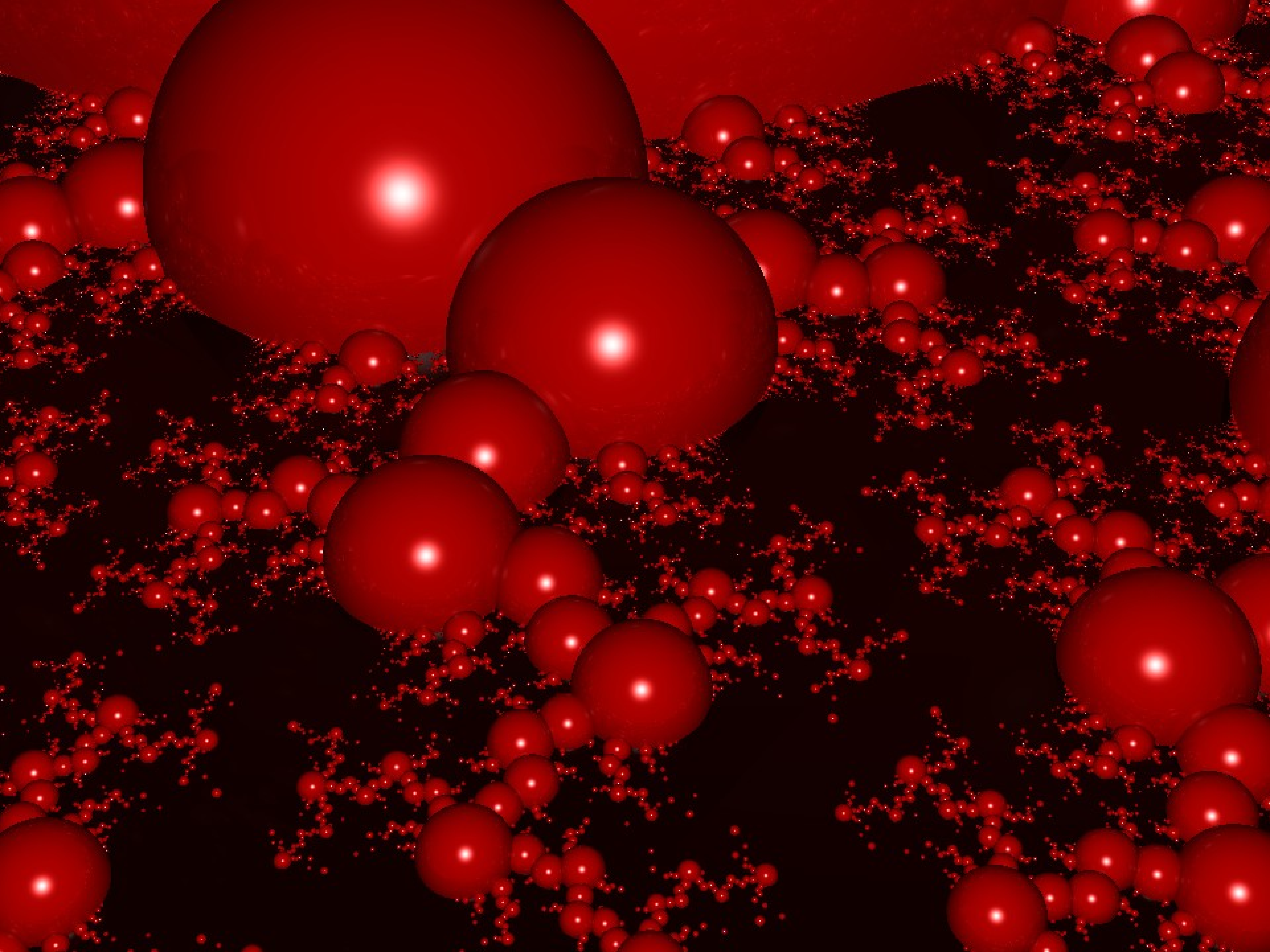
- Images: Jeff Brock, Nam-Gyu Kang, Scott Sheffield, and many others

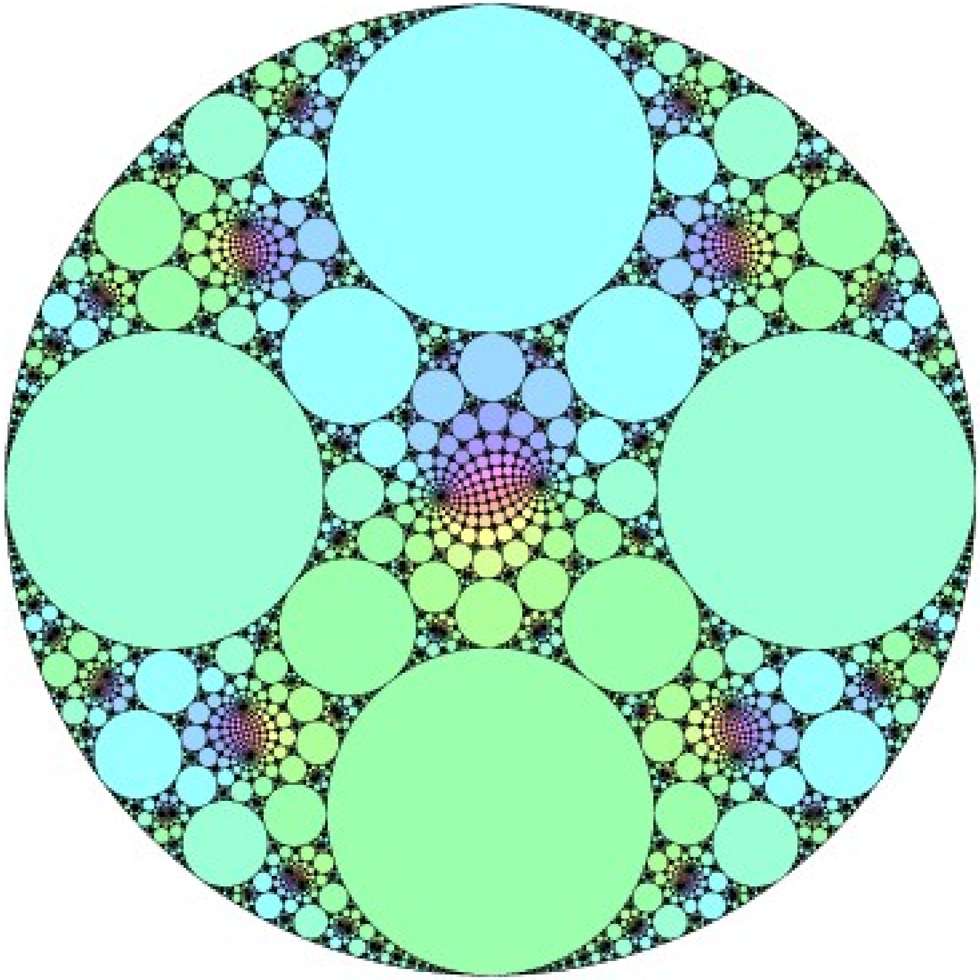
- Coauthors for theorem: K. Astala, A. Kupiainen, E. Saksman

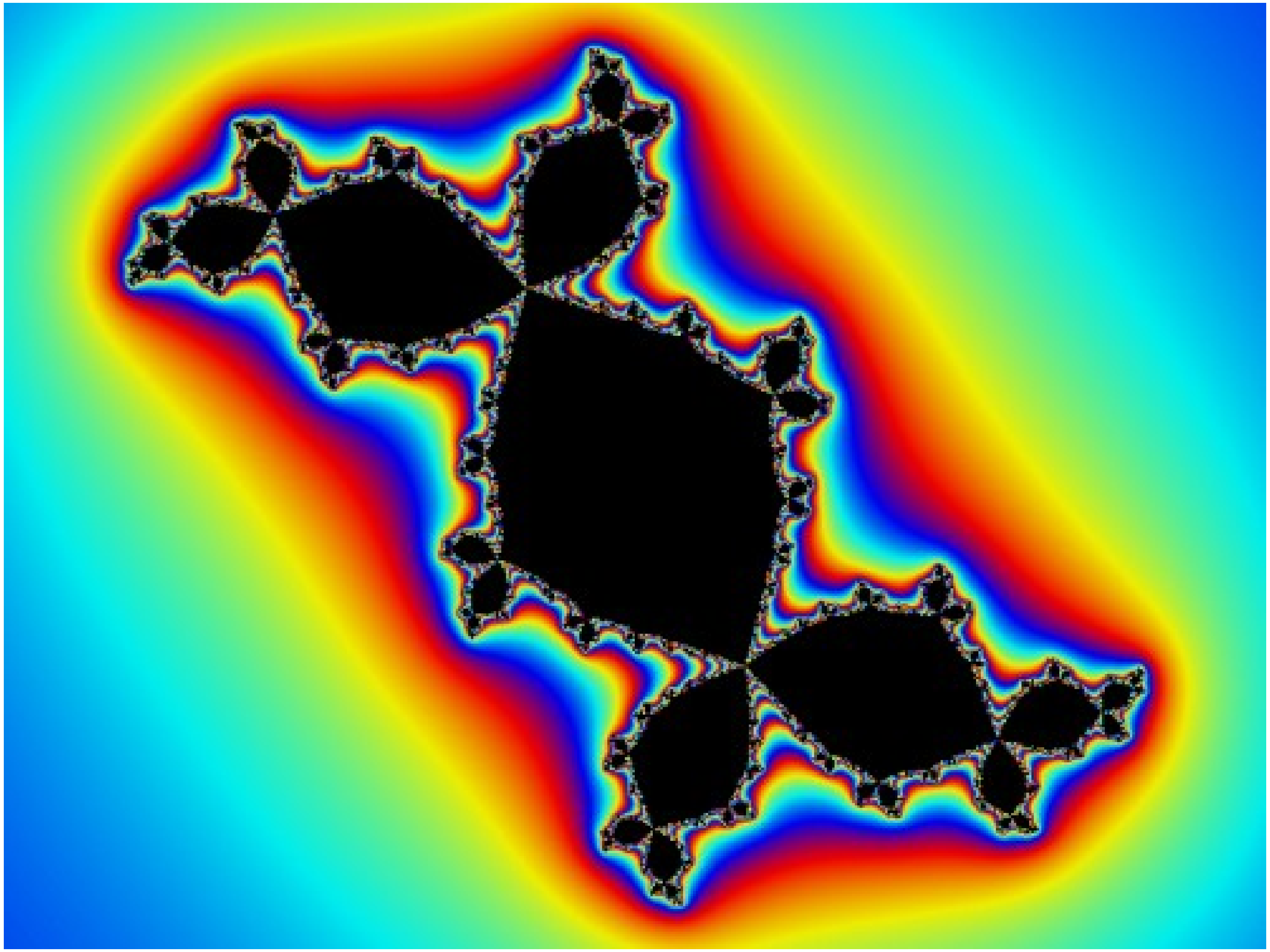
Objects under Discussion

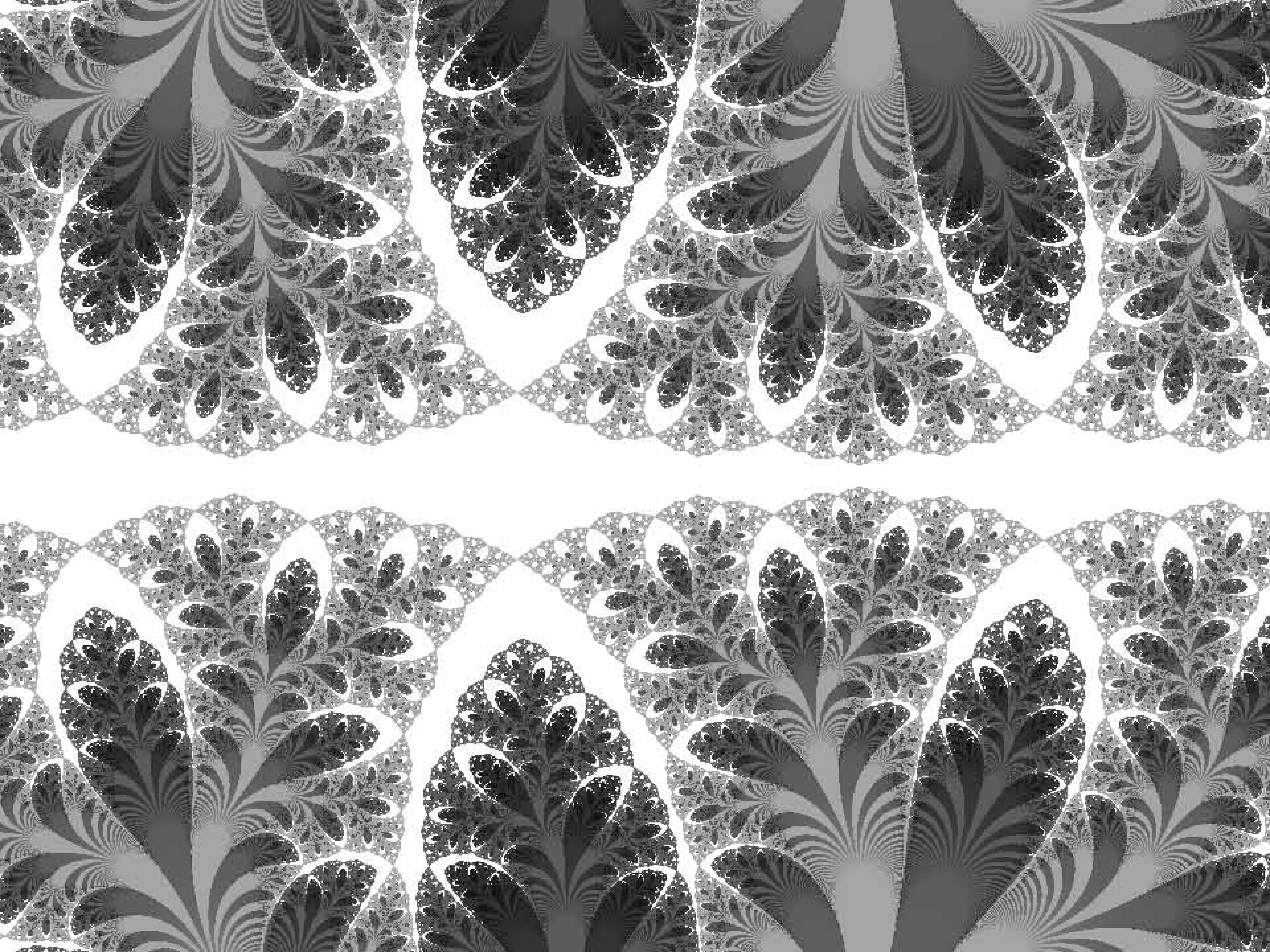
- Kleinian Groups
- Julia Sets
- SLE

We assume familiarity with the first two and give a brief exposition of SLE. The point is to understand the three objects from a translational point of view: a dictionary.









SLE

"Stochastic Loewner Evolution" or

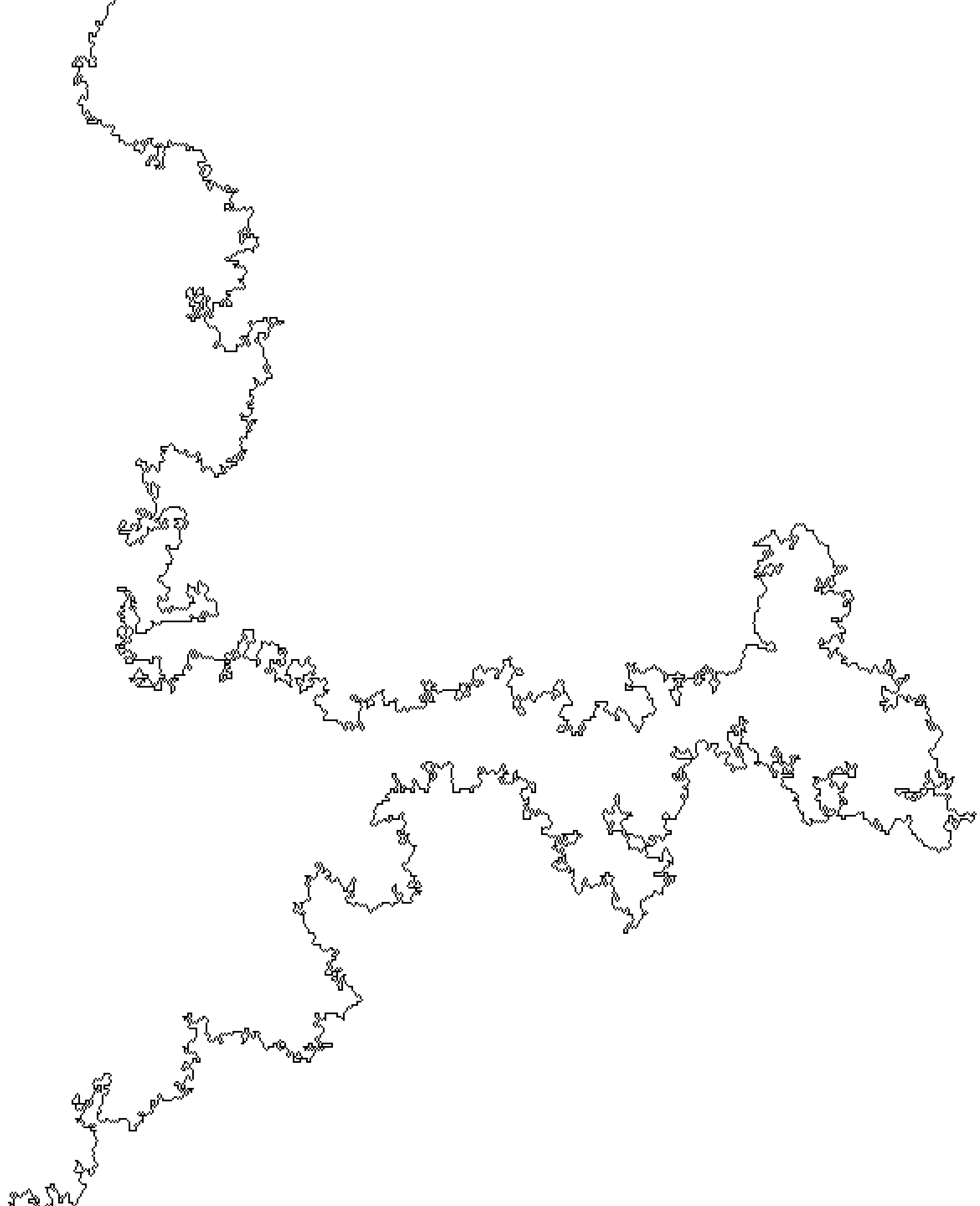
"Schramm – Loewner Evolution"

Oded Schramm (1961 – 2008)

Conformal Mappings from $\mathbb{H}_+$ to $\mathbb{H}_+$:

$$\partial_t F(t,z) = -2/(F(t,z) - B(\kappa t))$$

$$F(0,z) = z$$



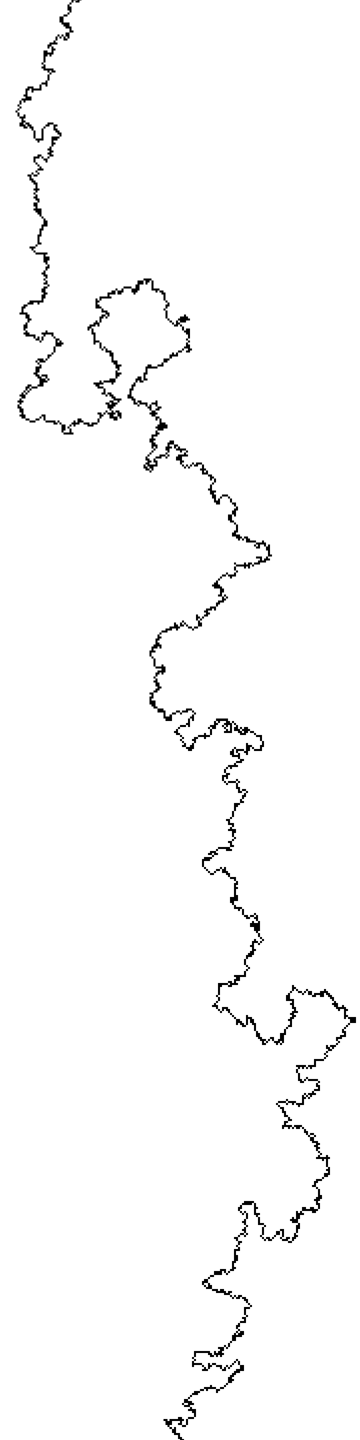
Provides a mapping from Wiener Space (Brownian trajectories on the line) to time dependent domains in the upper half space.

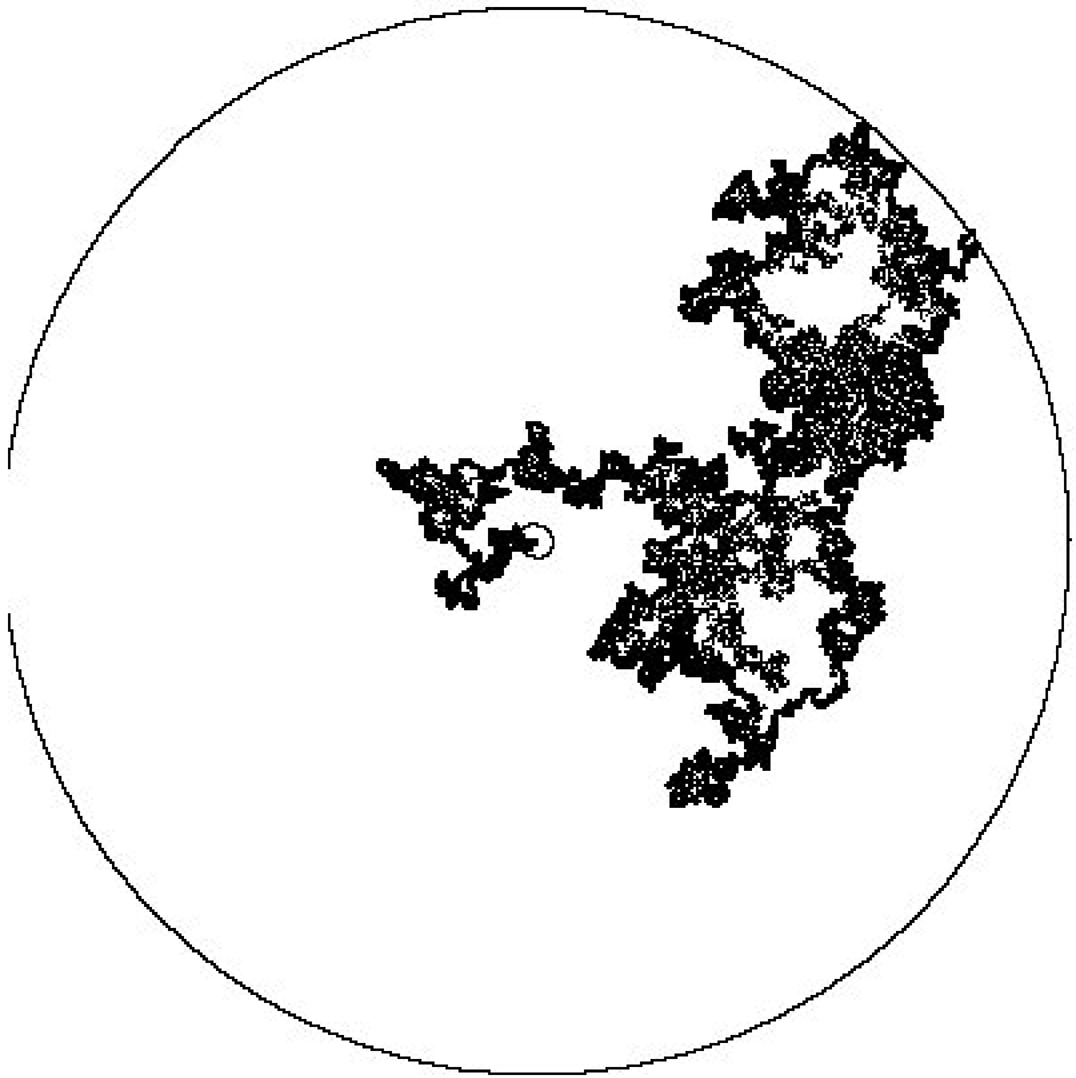
The domains in question are denoted by $\mathbb{H}_{+} \setminus K_t$. The "trace" at time t is the "curve-like" object K_t. It plays a fundamental role in Conformal Field Theory and for certain values of κ, corresponds to physical models. The parameter κ controls the "speed" of Brownian motion.

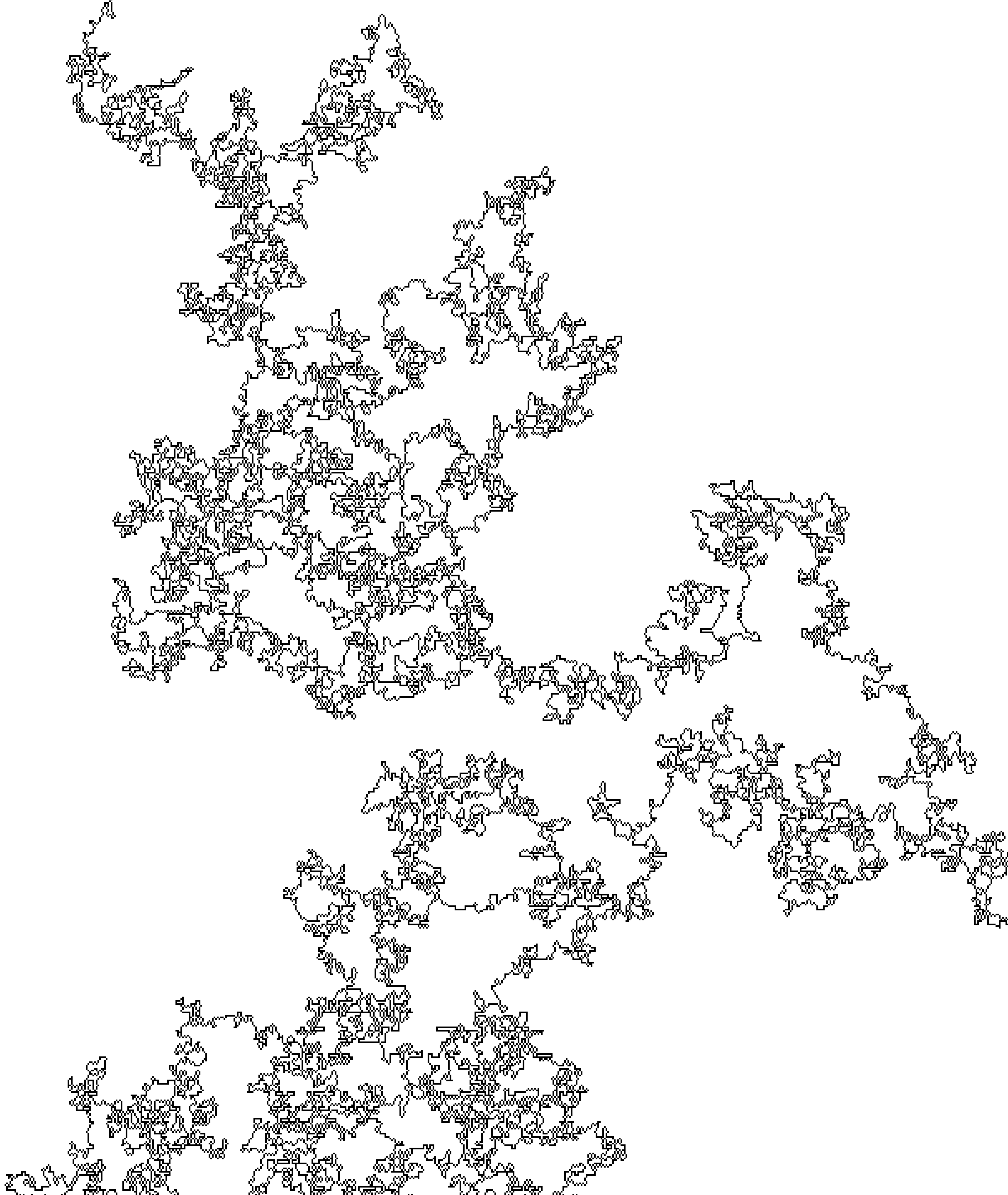
Some Values of Kappa

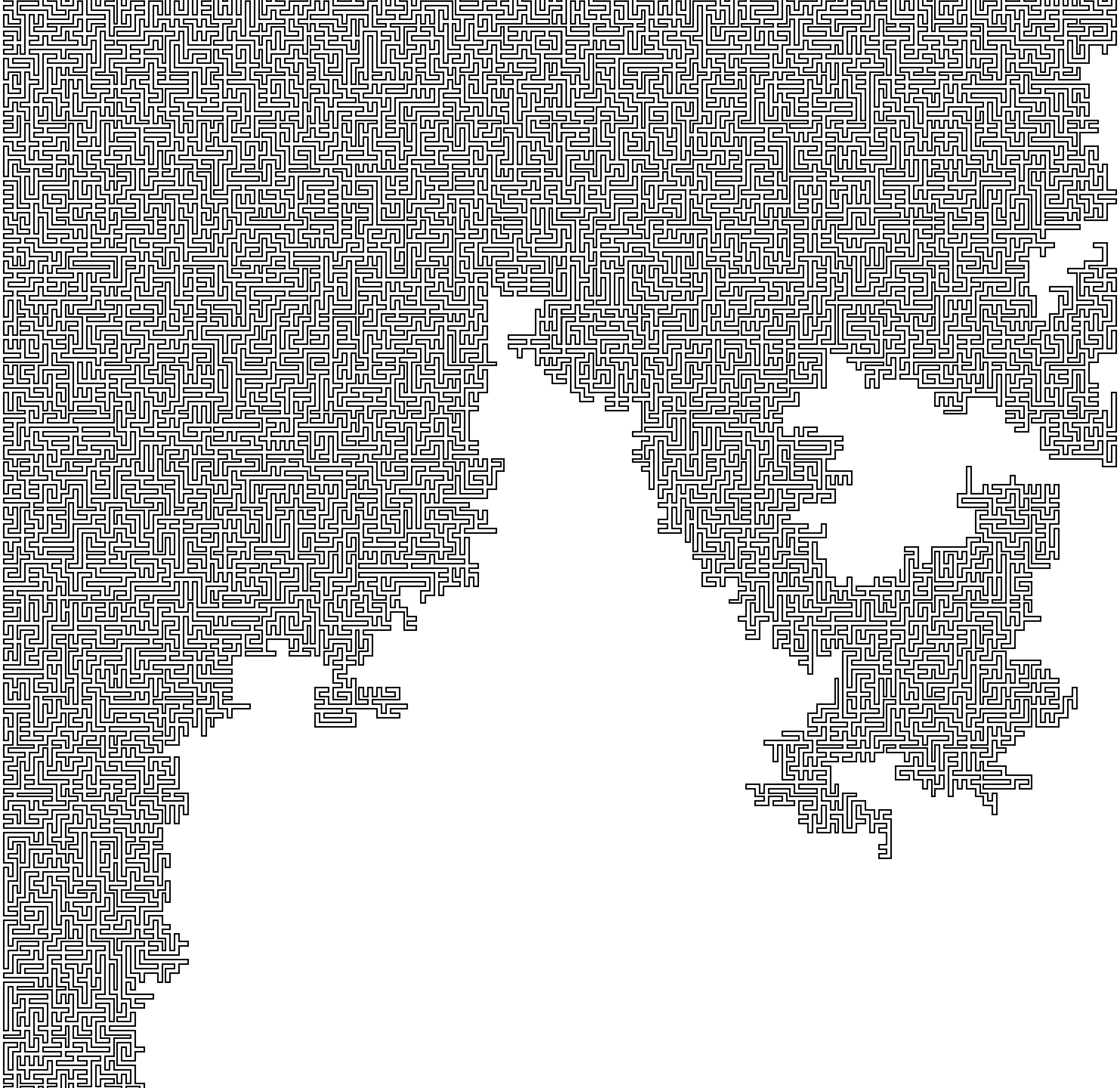
- $\kappa = 2$ LERW
- $\kappa = 8/3$ Brownian Frontier (Lawler-Schramm- Werner) Also conjectured to be “Self Avoiding Random Walk”
- = 4 “Harmonic Explorer”

  (Schramm -Sheffield)
- = 6 Percolation Clusters (Smirnov)
- = 8 Uniform Spanning Tree. Conjectured to be “Traveling Salesman”









The Sullivan Dictionary

The dictionary provides a way to translate results and proofs from one category to the other. The two “languages”:

2. Kleinian Groups (finitely generated)
3. Julia sets of Rational functions (Deg ≥ 2)

We will add today:

3. SLE

Examples of Dictionary Entries

Kleinian	Rational	SLE
Limit Set	Julia Set	SLE Trace
Ordinary Set	Fatou Domain	SLE Domain
Teichmüller Space	Mandelbrot Set	Wiener Space

Sample results

Kleinian Group: “Limit set Λ“

Corresponds to

Rational Function: “Julia set J”

Now add:

SLE: “Trace K_t “

Similarity: The natural closed set for SLE

Difference: The dynamics have time as parameter

Another example

- Kleinian: Ordinary set = $S^2 \setminus \Lambda$
- Rational: Fatou set = $S^2 \setminus J$
- SLE: Exterior of trace = $\mathbb{H}_+ \setminus K$

When $\kappa \leq 4$, K is a Jordan curve from 0 to infinity (a.s) so get two components. When $4 < \kappa < 8$ get (a.s.) infinitely many components

Parameter Space

- Kleinian: Teichmüller Space
- Rational: Mandelbrot Set
- SLE: Fix κ. Now the parameter space is "Wiener Space" = Ω (= [0,1], dω) and a parameter is a choice ω of a Brownian trajectory (at time κt): $B_\omega(\kappa t)$. Could also choose $\kappa^{1/2} B_\omega(t)$.

Inversion

- Kleinian: $z \rightarrow 1/z$ is distortionless, conjugation with group gives "same group"
- Rational ditto
- SLE: The Time Reversibility Problem: Does $z \rightarrow -1/z$ (Maps $\mathbb{H}_+$ to itself) map trace of SLE(κ) to trace of SLE(κ)? In other words, is this a measure preserving transformation?

Finiteness: Finite number of "domain classes"

- Kleinian: Ahlfors Finiteness Theorem for the ordinary set
- Rational: Sullivan's Non-wandering domains theorem for Fatou domains
- SLE: Statistical version: Whether two or infinitely many domains, they are "statistically similar on all scales".

X. Buff & A. Cheritat

Ahlfors conjecture = consequence of the works of Agol, Calegari, Gabai

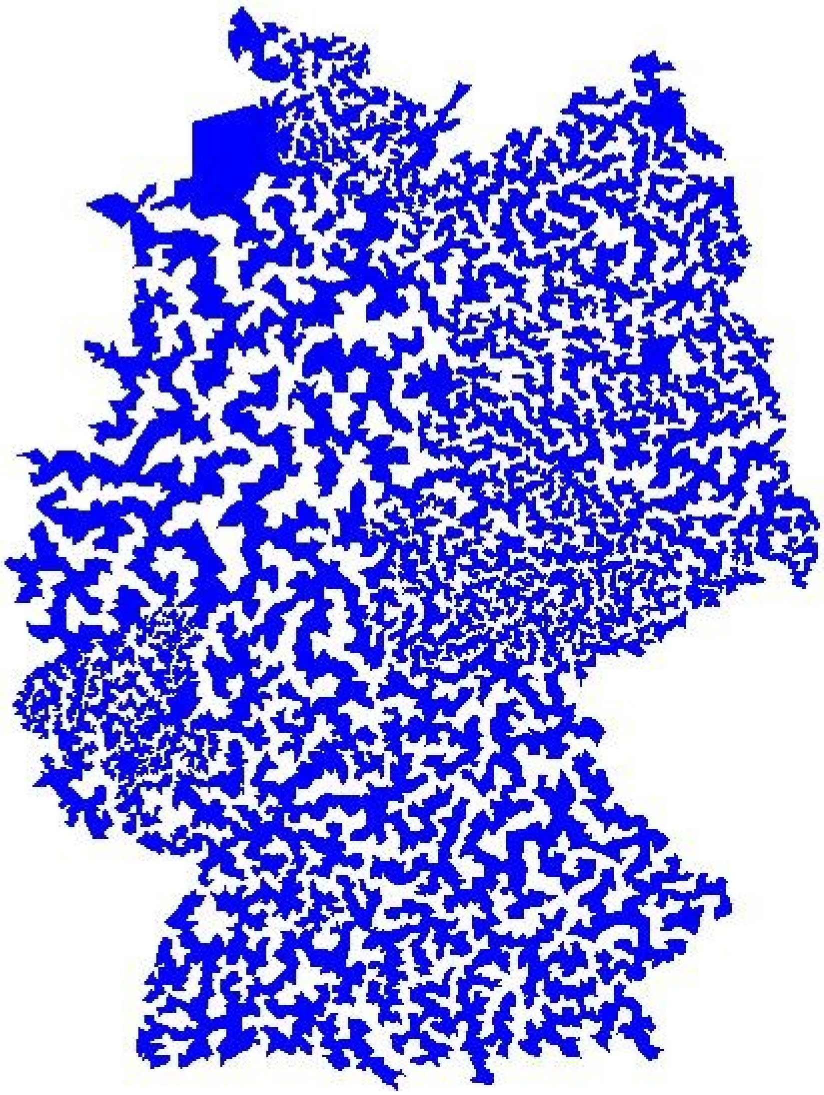
- Kleinian: $|\Lambda| = 0$ or 1? (Now verified)
- Rational: $|J| = 0$ or 1? (False!)

  Work of X. Buff & A. Cheritat provides counterexample

- SLE: $|K_{t=\infty}| = 0$ or 1? (True)

Theorem: $\mathrm{Dim}(K_t) = \max(1 + \kappa/8, 2)$ so measure = 0 unless $\kappa \geq 8$.

But still some mystery at $\kappa = 8$!



Positive Measure and Distortion

A vast oversimplification of why there can be quadratic Julia sets of positive measure, but no limit sets with $0 < |\Lambda| < 1$ on the sphere:

Polynomials can easily give large distortion, but Möbius transforms have zero distortion.

How about SLE? How much distortion is there, and what does it mean?

Geometry of the Components

Ordinary set: for example boundaries could be nice quasicircles.

Fatou domains: ditto (hyperbolic case)

SLE: DuPlantier Duality: the boundary of

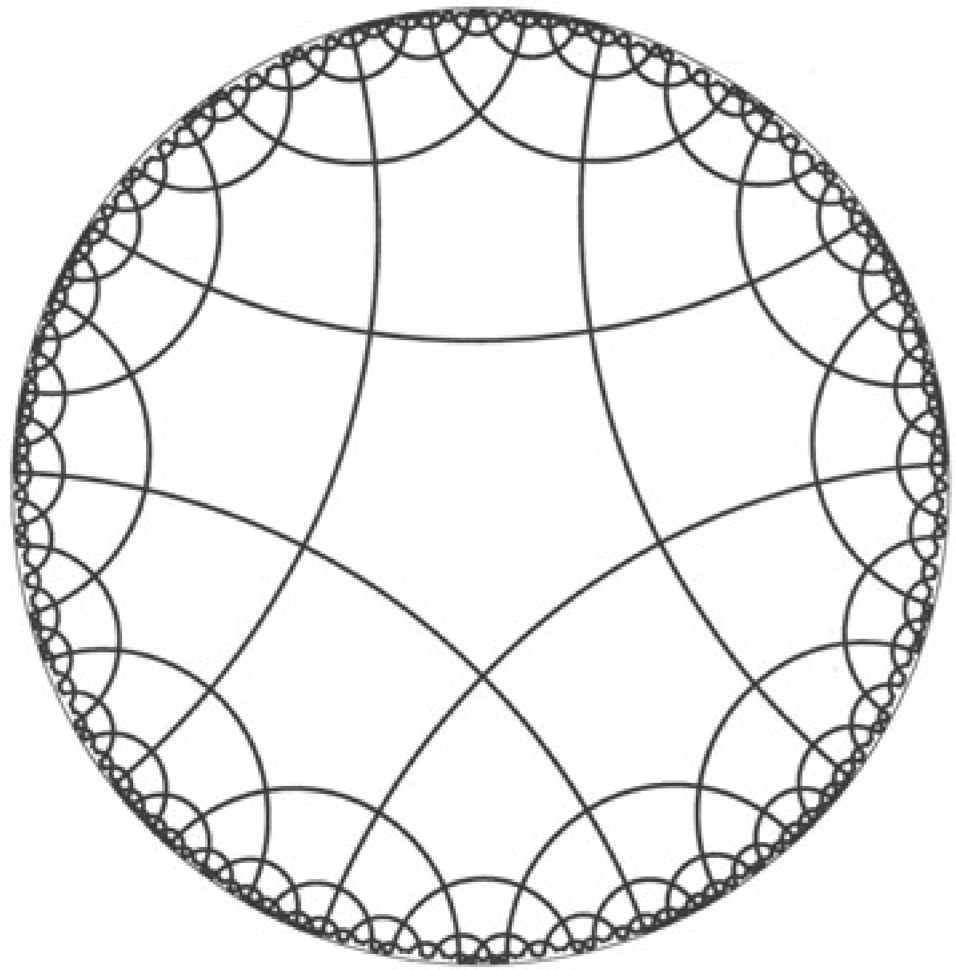
Trace SLE(κ) "=" Trace SLE(κ'), where

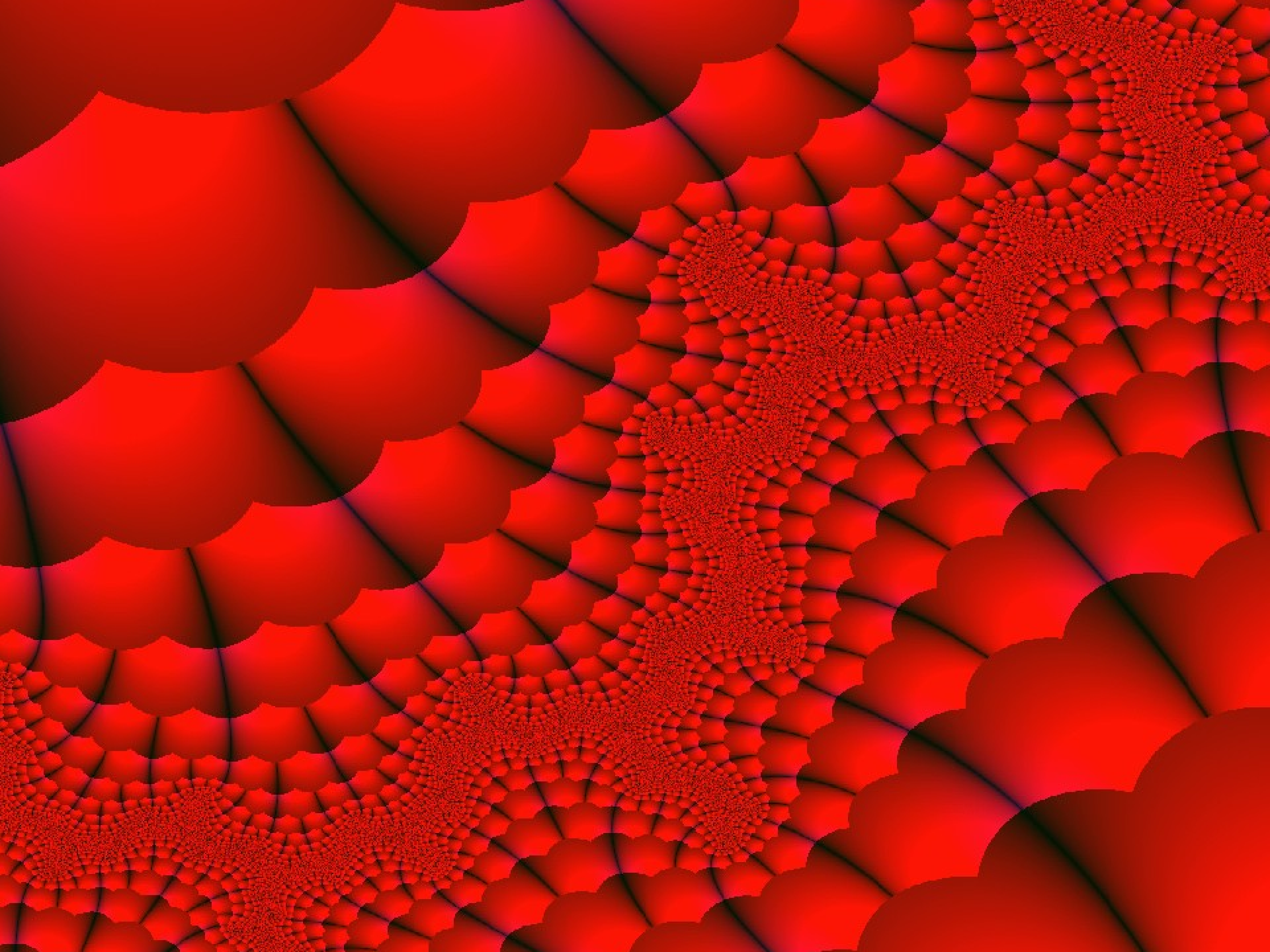
$$\kappa' = 16/\kappa$$

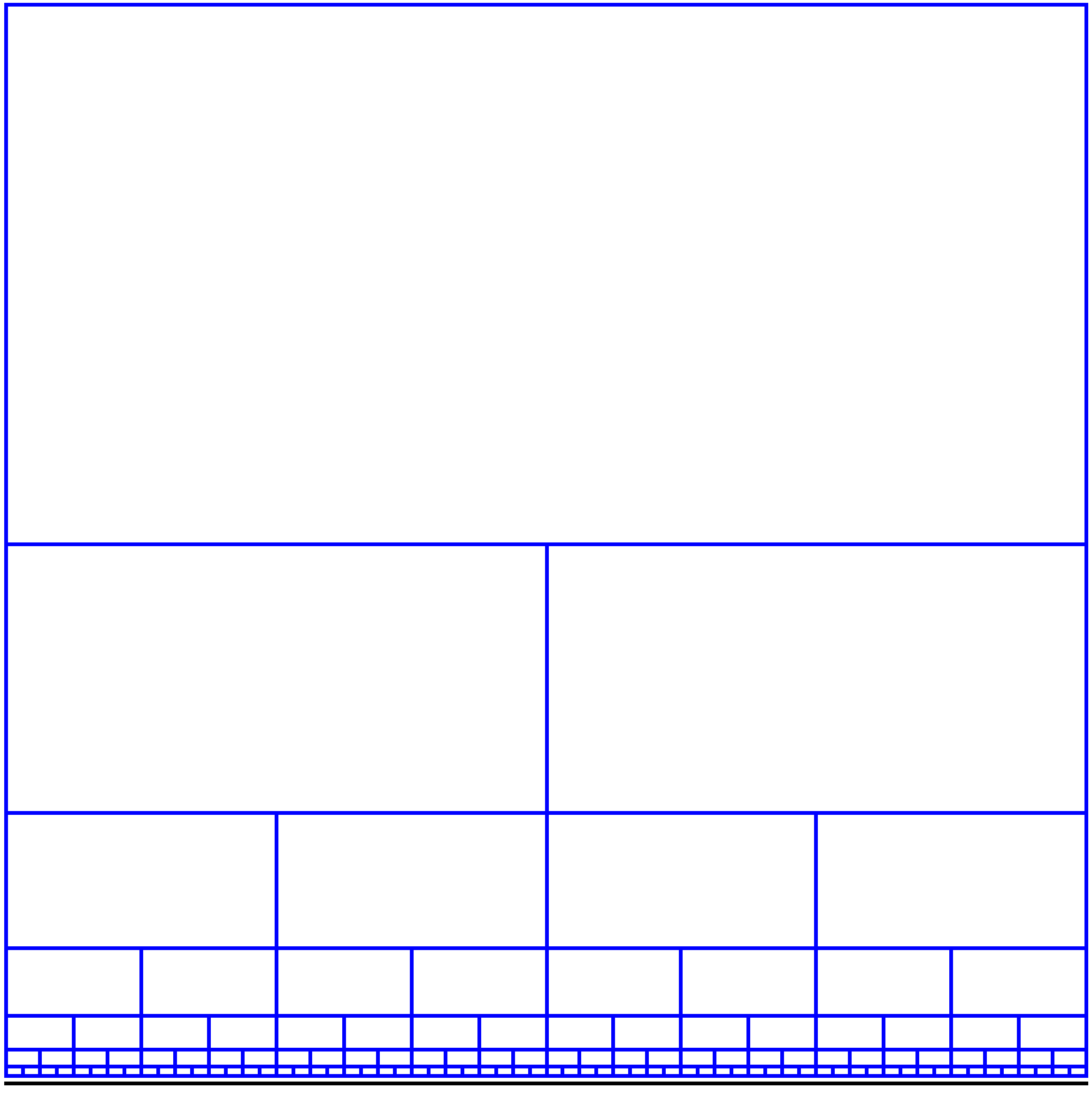
Geometry of boundaries seen from partitions arising from a “group”

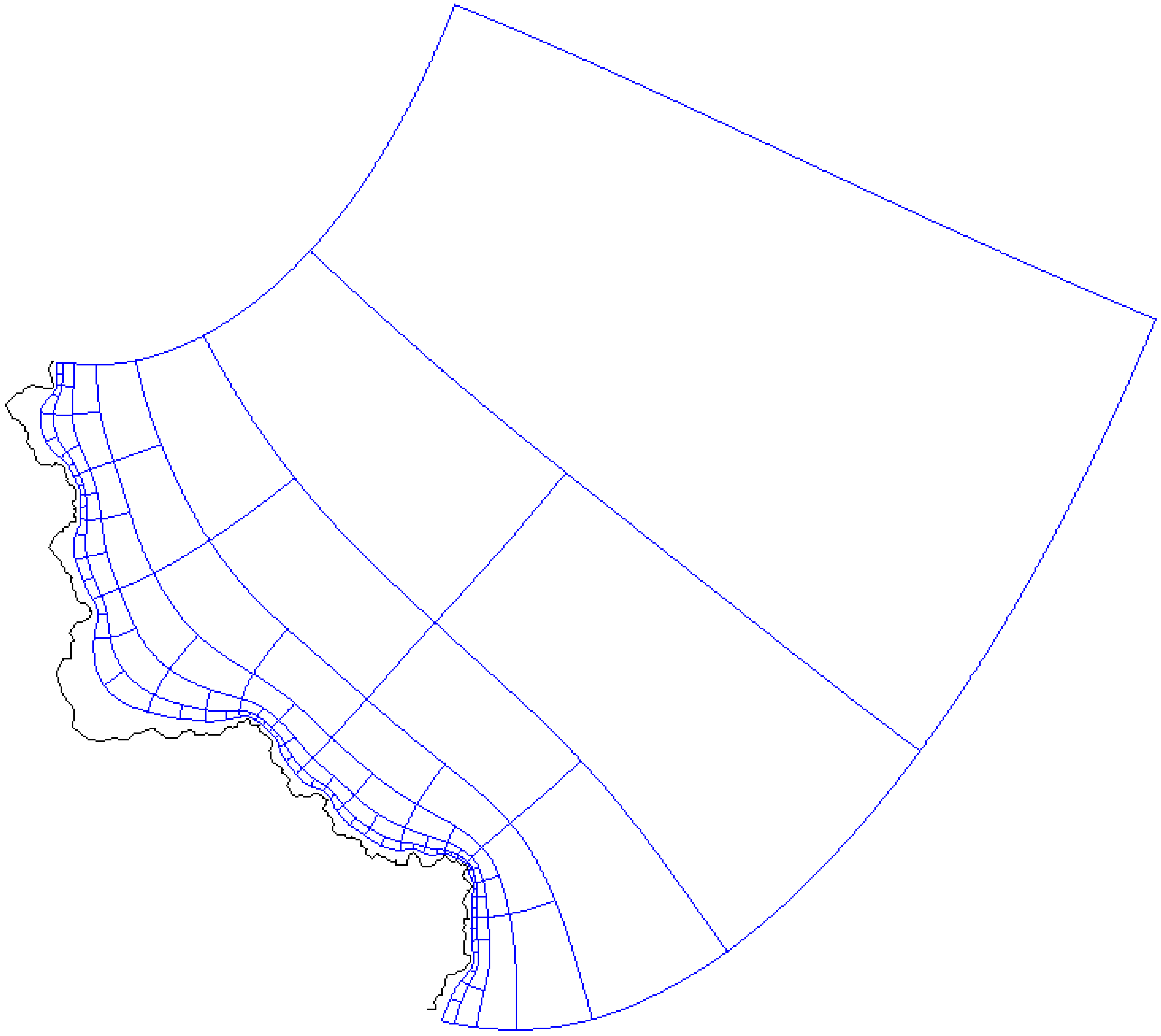
- All come from a hyperbolic lattice corresponding to the Whitney decomposition of the domain
- Kleinian: Fundamental domains, Orbit in a component of the ordinary set
- Rational: Markov partition, Inverse image of a point in a Fatou component
- SLE: Gaussian coefficients in the representation of Brownian motion and associated Whitney square.

(Now some partition pictures)







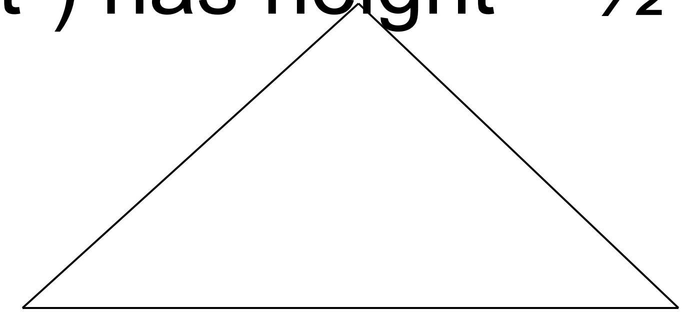


$$B_\omega(t) = \sum a_I(\omega) h_I(t)$$

where the $a_I(\omega)$ are i.i.d. Gaussian coefficients and

I is an interval = $[\, j2^{-k}, (j+1)2^{-k} \,]$, any $j,k \in \mathbb{Z}$.

$h_I(t)$ ("Tent") has height = ½ $\mathrm{Length}^{1/2}(I)$



More on complementary components for SLE

The complemenatary domains in SLE are NOT bounded by quasicircles, but are bounded by their statistical analogue:

Hölder Domains (Rohde – Schramm)

This means the Riemann mapping from the disk or upper half plane is a Hölder function.

Statistics contained in J, Makarov 1995

Reflection across the boundary

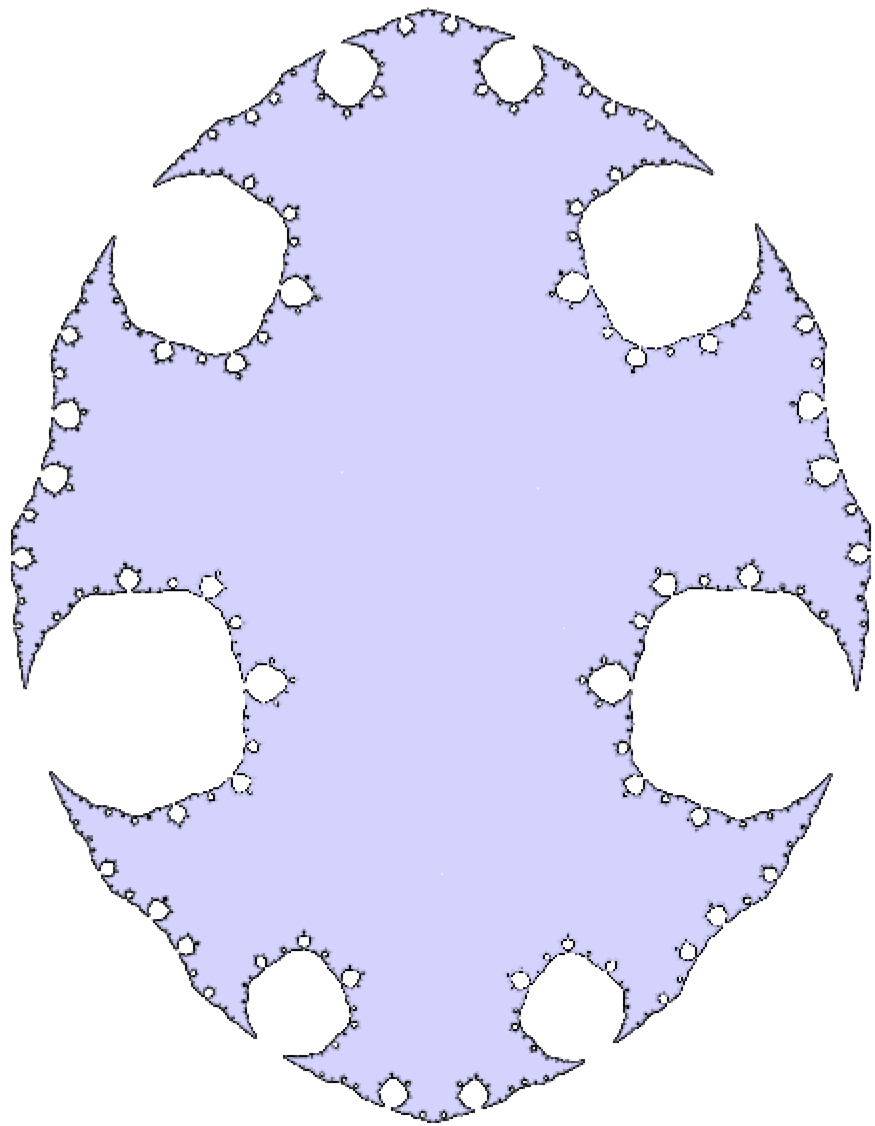
Example from the Kleinian world:

A Quasi-Fuchsian group

Then the limit set Λ is an Ahlfors quasicircle. The image of S^1 under a global QC mapping F:

$$\partial F \ = \ \mu(z)\overline{\partial} F \quad (\|\mu\|_\infty < 1)$$

Here the diff operators are 1/2 $(\partial_x -/+ i\partial_y)$



Exotic Homeos

Let $K \subset S^2$ be closed and let F be a homeo of S^2. Suppose further that F is holomorphic off K. We say F is **exotic** if F is not a Möbius transfom.

Exotic homeos are the potential problem in the Fatou conjecture for $z^2 + c$. If $|K| > 0$, it is easy to make an exotic homeo by solving Beltrami.

But that homeo might have no relation to dynamics.

Rigidity Examples

- Mostow Rigidity
- Rational (z^2 + c) There could be exotic homeos (w/o dynamical interpretation), but Central Conjecture in e.g. Fatou Conjecture, MLC is:

  Dynamic Rigidity Problem: If the homeo "conjugates" the dynamics on two z^2 + c dendrite Julia sets, are the polynomials the same?

SLE Analogue

For $\kappa < 4$ the trace is rigid. (No exotic homeos) This follows from Rohde – Schramm plus:

J – Smirnov: the boundary of a Hölder domain is rigid. (Sobolev Space Theorem)

N.G. Kang sharp result on Hölder exponent when $\kappa < 4$.

When $\kappa = 4$ not Hölder.

Distortion

- Kleinian and Rational: The usual method of deforming is by a Quasiconformal (QC) mapping.
- SLE: All such mappings are (a.s.)

  NOT QC but are so statistically. So

$$\| \mu \|_\infty = 1,$$

BUT: $| \mu(z) | < 1$ "often"

Lehto's Distortion Theorem

$$\int_0^1 \left(\int_0^{2\pi} (1 - |\mu(z + re^{i\theta})|)^{-1} \, d\theta \right)^{-1} dr = +\infty$$

implies there is a solution to Beltrami that is a global homeo. Proof: The integral gives bounds on moduli of annuli: they sum to $+\infty$.

Conformal Welding

Another tool in QC mapping, arises in simultaneous uniformization.

Let D be a s.c. domain with Jordan curve Γ as boundary. Let F = Riemann map from $\{|z| < 1\}$ to inside domain, G from $\{|z| > 1\}$ to outside domain. Get homeomorphism

$$\Phi = G^{-1} \circ F : S^1 \to S^1$$

Another Dictionary Entry?

- Kleinian: Quasi-Fuchsian Group
- Rational: Quasi-circle Julia Set
- SLE: For $0 < \kappa < 4$ can one make closed "SLE(κ) loops"? (Via conformal welding?)

We know they must be Hölder domains, and if they come from conformal welding, they must be "rigid".

A Theorem with a Problem

(Work of Astala, J, Kupiainen, Saksman to appear soon!)

Exponentiate the Gaussian Free Field

(= Massless Free Field) after subtracting

infinity. Multiply by constant to get derivative

of homeo(S^1).

Theorem: The homeo is (a.s.) a welding curve and the curve is "rigid" (unique).

But is it "Renormalized SLE Loops" for $\kappa < 4$?

Σ = sum over all j >0 of

$$X_j(\omega)\, j^{-1/2}\cos(j\theta) + X'_j(\omega)\, j^{-1/2}\sin(j\theta) - 1/2j$$

Here X_j , X'_j = i.i.d. Brownian motion at time t corresponding to kappa)

$$\Phi'(\theta) = \text{const}(\omega)\, e^{\Sigma}$$

Then a.s. Φ is welding with uniqueness for the welding curve (rigid)

The next slide shows the price and volume charts of the Sara Lee Corp. on May 1, 2009. (The ticker symbol is SLE, and note that the trading day ends at 4.) The driving force for SLE is Brownian motion (with a parameter kappa) and this is in a sense approximately what the price chart loolk like. The volume chart mimics the multifractal measures in the theorem.


SLE
1-May 3:59pm
8.45
8.40
8.35
8.30
8.25
8.20
8.15
10am
12pm
2pm
4pm
150.00
100.00
50.00
0.00
Thousands
Volume
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A Very Short Bibliography

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Lawler, Gregory F.; Schramm, Oded; Werner, Wendelin (2001), "The dimension of the planar Brownian frontier is 4/3", *Mathematical Research Letters* **8** (4): 401–411

Lawler Schramm werner Acta Math., 2001;Rohde Schramm Annal of Math 2005

Schramm, Oded (2000), "Scaling limits of loop-erased random walks and uniform s", *Israel Journal of Mathematics* **118**: 221–288, doi: 10.1007/BF02803524, MR1776084, http://arxiv.org/abs/math.PR/9904022 Schramm's original paper, introducing SLE