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Noncommutative Geometry and Number Theory

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Contents

Introduction

In almost every branch of mathematics we use the ring of rational integers, yet in looking beyond the formal structure of this ring we often encounter great gaps in our understanding. The need to find new insights into the ring of integers is, in particular, brought home to us by our inability to decide the validity of the classical Riemann hypothesis, which can be thought of as a question on the distribution of prime numbers. Inspired by ideas from noncommutative geometry, Alain Connes [**8**], [**10**], [**9**] has in recent years proposed a set-up within which to approach the Riemann Hypothesis. The following chapters provide an introduction to these ideas of Alain Connes and are intended to aid in a serious study of his papers and in the analysis of the details of his proofs, which for the most part we do not reproduce here. We also avoid reproducing too much of the classical material, and choose instead to survey, without proofs, basic facts about the Riemann Hypothesis needed directly for understanding Connes's papers. These chapters should be read therefore with a standard textbook on the Riemann zeta function at hand—for example, the book of Harold M. Edwards [**15**], which also includes a translation

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of Riemann's original paper. For the function field case, the reader can consult André Weil's book [**39**]. A good introduction to the Riemann zeta function and the function field case can also be found in Samuel J. Patterson's study [**30**]. A concise and informative survey of the Riemann Hypothesis, from which we quote several times, is given by Enrico Bombieri on the Clay Mathematics Institute website [**3**] (see also the updated report of Peter Sarnak on that same website). Some advanced notions from number theory are referred to as motivation for Connes's approach, but little knowledge of number theory is assumed for the discussion of the results of his papers. Although Connes's papers apply to arbitrary global fields, we most often restrict our attention to the field of rational numbers, as this still brings out the main points and limits the technicalities.

There are some similarities between Alain Connes's work in [**8**], [**10**], [**9**] and work of Shai Haran in [**21**], [**23**], [**22**]. We do not pursue here the relation to Shai Haran's papers, although we refer to them several times.

1. The objects of study

1.1. The Riemann zeta function. Riemann formulated his famous hypothesis in 1859 in a foundational paper [**31**], just 8 pages in length, on the number of primes less than a given magnitude. The paper centers on the study of a function $\zeta(s)$, now called the Riemann zeta function, which has the formal expression,

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}, \tag{1}$$

the right hand side of which converges for $\Re(s) > 1$. This function, in fact, predates Riemann. In a paper [**17**], published in 1748, Euler observed a connection with primes via the formal product expansion,

$$\sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{p \text{ prime}} (1 - p^{-s})^{-1}, \tag{2}$$

valid for $\Re(s) > 1$. Equation (2) is a direct result of the unique factorization, up to permutation of factors, of a positive rational integer into a product of prime numbers. The contents of Euler's paper are described in [**15**]. It seems that Euler was aware of the asymptotic formula,

$$\sum_{p<x} \frac{1}{p} \sim \log(\log x), \qquad (x \to \infty)$$

where the sum on the left hand side is over the primes p less than the real number x.

The additive structure of the integers leads to considering negative as well as positive integers and to the definition of the usual absolute value $|\cdot|$ on the ring $\mathbb{Z}$ of rational integers, defined by

$$|n| = \mathrm{sg}(n)\, n, \qquad n \in \mathbb{Z}.$$

The p-adic valuations are already implicit in the unique factorization of positive integers into primes. Namely, for every prime p and every integer n one can write

$$n = p^{\mathrm{ord}_p(n)} n'$$

where n' is an integer not divisible by p. The p-adic absolute value of n is then defined to be

$$|n|_p = p^{-\mathrm{ord}_p(n)}.$$

We denote by $\mathbb{Q}$ the field of fractions of $\mathbb{Z}$, namely the field of rational numbers, and by $M_{\mathbb{Q}}$ the set of valuations just introduced, extended to $\mathbb{Q}$ in the obvious way, and indexed by ∞ and by the primes p. We write, for $x \in \mathbb{Q}$,

$$|x|_v = |x|, \qquad v = \infty \in M_{\mathbb{Q}}$$

and

$$|x|_v = |x|_p, \qquad v = p \in M_{\mathbb{Q}}.$$

The following important observation is obvious from the definitions.

Product Formula: *For every $x \in \mathbb{Q}$, $x \neq 0$, we have*

$$\prod_{v \in M_{\mathbb{Q}}} |x|_v = 1.$$

Riemann derived a formula for $\sum n^{-s}$ valid for all $s \in \mathbb{C}$. For $\Re(s) > 0$, the Γ-function is defined by

$$\Gamma(s) = \int_0^\infty e^{-x} x^{s-1} dx.$$

The function $\Gamma(s)$ has an analytic continuation to all $s \in \mathbb{C}$ with simple poles at $s = 0, -1, -2, \ldots$, with residue $(-1)^m m!$ at $-m$, $m \geq 0$. This can be seen using the formula

$$\Gamma(s) = \lim_{N \to \infty} \frac{1 \cdot 2 \cdot \cdots \cdot N}{s(s+1) \cdots (s+N-1)} (N+1)^{s-1}.$$

Moreover, we have $s\Gamma(s) = \Gamma(s+1)$, and at the positive integers $m > 0$, we have $\Gamma(m) = (m-1)!$. Riemann observed that, for $\Re(s) > 1$,

$$\Gamma(\frac{s}{2}) \pi^{-s/2} \sum_{n=1}^\infty n^{-s} = \int_0^\infty \sum_{n=1}^\infty \exp(-n^2 \pi x) x^{s/2} \frac{dx}{x}. \tag{3}$$

Moreover, he noticed that the function on the right hand side is unchanged by the substitution $s \mapsto 1 - s$ and that one may rewrite the integral in (3) as

$$\int_1^\infty \sum_{n=1}^\infty \exp(-n^2 \pi x) \left(x^{s/2} + x^{(1-s)/2} \right) \frac{dx}{x} \quad - \quad \frac{1}{s(1-s)}, \tag{4}$$

which converges for all $s \in \mathbb{C}$ and has simple poles at $s = 1$ and $s = 0$. This shows that $\zeta(s) = \sum n^{-s}$, $\Re(s) > 1$, can be analytically continued to a function $\zeta(s)$ on all of $s \in \mathbb{C}$ with a simple pole at $s = 1$ (the pole at $s = 0$ in (4) being accounted for by $\Gamma(\frac{s}{2})$).

Riemann defined, for $t \in \mathbb{C}$ given by $s = \frac{1}{2} + it$, the function

$$\xi(t) = \frac{1}{2} s(s-1) \Gamma(\frac{s}{2}) \pi^{-s/2} \zeta(s), \tag{5}$$

for which we have the following important result.

Theorem 1. (i) *Let Z be the set of zeros of $\xi(t)$. We have a product expansion of the form*

$$\xi(t) = \frac{1}{2} \pi^{-s/2} e^{bs} \prod_{\rho \in Z} (1 - \frac{s}{\rho}) e^{s/\rho}, \qquad s = \frac{1}{2} + it, \tag{6}$$