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The Riemann Hypothesis: Arithmetic and Geometry

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1. Introduction

This paper describes basic properties of the Riemann zeta function and its generalizations, presents various formulations of the Riemann hypothesis, and indicates various geometric analogies. It briefly discusses the approach of A. Connes to a "spectral" interpretation of the Riemann zeros via noncommuative geometry, which is treated in detail by Paula Tretkoff [**33**] in this volume.

The origin of the Riemann hypothesis was as an arithmetic question concerning the asymptotic distribution of prime numbers. In the last century profound geometric analogues were discovered, and some of them proved. In particular there are striking analogies in the subject of spectral geometry, which is the study of global geometric properties of a manifold encoded in the eigenvalues of various geometrically natural operators acting on functions on the manifold. This has led to the search for a "geometric" and/or "spectral" interpretation of the zeros of the Riemann zeta function.

One should note that a geometric or spectral interpretation of the zeta zeros by itself is not enough to prove the Riemann hypothesis; the essence of the problem seems to lie in a suitable "positivity property" which must be established. A hope

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is that there exists such an interpretation in which the positivity will be a natural (and provable) consequence of the internal structure of the "geometric" object.

2. Basics

The Riemann zeta function is an analytic device that encodes information about the ring of integers $\mathbb{Z}$. In particular, it relates to the multiplicative action of $\mathbb{Z}$ on the additive group $\mathbb{Z}$. In its most elementary form, the Riemann zeta function can be defined by the well-known series

$$\zeta(s) = \sum_{n=1}^{\infty} n^{-s},$$

where the domain of convergence is the half-plane $\{s : \Re(s) > 1\}$. This series was studied well before Riemann, and in particular Euler observed that it can be rewritten in the product form

$$\begin{aligned}\zeta(s) &= \prod_{p \text{ prime}} (1 + p^{-s} + p^{-2s} + \dots) \\ &= \prod_{p \text{ prime}} (1 - p^{-s})^{-1}.\end{aligned}$$

The zeta function can be extended to a meromorphic function on the entire complex plane. More specifically, if we define the completed zeta function $\hat{\zeta}(s)$ by

$$\hat{\zeta}(s) := \pi^{-\frac{s}{2}}\Gamma(\frac{s}{2})\zeta(s),$$

then we have the following.

Theorem 2.1. *The completed zeta function $\hat{\zeta}$ has an analytic continuation to the entire complex plane except for simple poles at $s = 0, 1$. Furthermore, this function $\hat{\zeta}$ satisfies the functional equation*

$$\hat{\zeta}(s) = \hat{\zeta}(1 - s).$$

Proof. With a suitable change of variables, the integral definition of Γ gives

$$\Gamma(\frac{s}{2}) = n^s \pi^{\frac{s}{2}} \int_0^{\infty} e^{-\pi n^2 x} x^{\frac{s}{2}-1} dx, \tag{1}$$

for every $n \in \mathbb{Z}^+$. Rearranging (1) and summing over $n \in \mathbb{Z}^+$, one can show that for all $s \in \mathbb{C}$ with $\Re(s) > 1$,

$$\begin{aligned}\hat{\zeta}(s) = \pi^{-\frac{s}{2}}\Gamma(\frac{s}{2})\zeta(s) &= \int_0^{\infty} \sum_{n=1}^{\infty} e^{-\pi n^2 x} x^{\frac{s}{2}-1} dx \\ &= \frac{1}{2}\int_0^{\infty} (\theta(x) - 1) x^{\frac{s}{2}} \frac{dx}{x}, \qquad (2)\end{aligned}$$

where $\theta(x) = \sum_{n \in \mathbb{Z}} e^{-\pi n^2 x}$. This θ-function satisfies the functional equation

$$\theta(x^{-1}) = \sqrt{x}\theta(x).$$

Now, the integral in (2) can be split as

$$\int_0^{\infty} (\theta(x) - 1) x^{\frac{s}{2}} \frac{dx}{x} = \int_0^{1} (\theta(x) - 1) x^{\frac{s}{2}} \frac{dx}{x} + \int_1^{\infty} (\theta(x) - 1) x^{\frac{s}{2}} \frac{dx}{x}.$$

Applying the change of variables $x \mapsto x^{-1}$ in the first of these, we obtain

$$\hat{\zeta}(s) = \frac{1}{s(s-1)} + \frac{1}{2}\int_1^\infty (\theta(x)-1)(x^{\frac{1-s}{2}} + x^{\frac{s}{2}})\frac{dx}{x}. \tag{3}$$

This integral is uniformly convergent on $\{s : \Re(s) > \sigma\}$ for any $\sigma \in \mathbb{R}$, and thus is an entire function of s. Therefore, (3) exhibits the meromorphic continuation of $\hat{\zeta}$, and it clearly satisfies the functional equation.

□

It is now natural to define the entire function

$$\xi(s) = \frac{1}{2}s(s-1)\hat{\zeta}(s).$$

The factor of $\frac{1}{2}$ here was introduced by Riemann and has stuck. Hadamard showed that ξ has the product expansion

$$\xi(s) = \prod_\rho (1 - \frac{s}{\rho})e^{\frac{s}{\rho}},$$

where the product is over the zeros of ξ.

The location of the zeros of ξ is of great importance in number theoretic applications of the zeta function. Euler's product formula easily shows that every zero ρ has $\Re(\rho) \leq 1$, and the functional equation then gives that all zeros lie in the closed strip $\{s : 0 \leq \Re(\rho) \leq 1\}$. In fact, it can be shown that all zeros lie within the open strip $\{s : 0 < \Re(\rho) < 1\}$, although this is a non-trivial result.

Since $\xi(s)$ is real-valued for real values of s, it is clear that we have

$$\xi(\bar{s}) = \overline{\xi(s)}.$$

Thus if ρ is a zero of ξ, so are $\bar{\rho}$, $1-\rho$ and $1-\bar{\rho}$. Consequently, zeros on the line $\Re(s) = \frac{1}{2}$ occur in conjugate pairs, and zeros off this line occur in quadruples.

The Riemann hypothesis is now stated simply as follows.

Conjecture. *All zeros of $\xi(s)$ lie on the line $\Re(s) = \frac{1}{2}$.*

Riemann confirmed the position of many of the zeros of $\xi(s)$ to be on this critical line by hand, by making use of the symmetry from the functional equation. For if the approximate location of a zero close to the critical line is known, one can consider a small contour C around the zero which is symmetric about the critical line. By estimating the integral

$$\frac{1}{2\pi \mathrm{i}}\int_C \frac{\xi'(s)}{\xi(s)}ds,$$

one can determine the number of zeros (including multiplicity) enclosed within the curve C. If only one such zero exists, symmetry dictates that it must lie on the critical line. To date, no double zeros have been found on the critical line.

The Riemann hypothesis can be reformulated in a number theoretic context as follows. If we define

$$\pi(x) = \sum_{\substack{p \leq x \\ p \text{ prime}}} 1$$

as usual, then the Riemann hypothesis is known to be equivalent to the veracity of the following error term in the Prime Number Theorem:

$$\pi(x) = \int_2^x \frac{dt}{\log t} + O(x^{\frac{1}{2}}(\log x)^2).$$