Clay Mathematics Proceedings
Volume 6, 2006

On Novikov-Type Conjectures

Stanley S. Chang and Shmuel Weinberger

Contents

This paper is based on lectures given by the second author at the 2000 Summer conference at Mt. Holyoke. We also added a brief epilogue, essentially "What there wasn't time for." Although the focus of the conference was on noncommutative geometry, the topic discussed was conventional commutative motivations for the circle of ideas related to the Novikov and Baum-Connes conjectures. While the article is mainly expository, we present here a few new results (due to the two of us).

It is interesting to note that while the period from the 80's through the mid-90's has shown a remarkable convergence between index theory and surgery theory (or more generally, the classification of manifolds) largely motivated by the Novikov conjecture, most recently, a number of divergences has arisen. Possibly, these subjects are now diverging, but it also seems plausible that we are only now close to discovering truly deep phenomena and that the difference between these subjects is just one of these. Our belief is that, even after decades of mining this vein, the gold is not yet all gone.

As the reader might guess from the title, the focus of these notes is not quite on the Novikov conjecture itself, but rather on a collection of problems that are suggested by heuristics, analogies and careful consideration of consequences. Many of the related conjectures are false, or, as far as we know, not directly mathematically related to the original conjecture; this is a good thing: we learn about the subtleties of the original problem, the boundaries of the associated phenomenon, and get to learn about other realms of mathematics.

2000 *Mathematics Subject Classification.* Primary 57R20; Secondary 01A60, 46L99, 57R67, 58G12.

Of course in such a subject of active research, there are necessarily many interesting developments in this field since the 2000 conference and the subsequent drafting of this article. We regret that we cannot incorporate all of these newer findings in this document.

1. Topology and K-theory

1.1. For the topologist, the Novikov conjecture is deeply embedded in one of the central projects of his field, that of classifying manifolds within a homotopy type up to homeomorphism or diffeomorphism. To put matters in perspective, let us begin by reviewing some early observations regarding this problem.

The first quite nontrivial point is that there are closed manifolds that are homotopy equivalent but not diffeomorphic (homeomorphism is much more difficult). It is quite easy to give examples which are manifolds with boundary: the punctured torus and the thrice punctured 2-sphere are homotopy equivalent but not diffeomorphic; their boundaries have different numbers of components.

The first class of examples without boundary are the lens spaces: quotients of the sphere by finite cyclic groups of isometries of the round metric. To be concrete, let S^{2n-1} be the unit sphere in $\mathbb{C}^n$ with coordinates $(u_1, \ldots, u_n)$. For any n-tuple of primitive k-th roots of unity $e^{2\pi i a_r/k}$, one has a $\mathbb{Z}_k$ action by multiplying the r-th coordinate by the r-th root of unity. The quotient manifolds under these actions are homotopy equivalent (preserving the identification of fundamental group with $\mathbb{Z}_k$) iff the products of the rotation numbers $a_1 a_2 \cdots a_n$ are the same mod k. On the other hand, these manifolds are diffeomorphic iff they are isometric iff the sets of rotation numbers are the same (*i.e.* they agree after reordering). There are essentially two different proofs of this fact, both of which depend on the same sophisticated number-theoretic fact, the Franz independence lemma.

The first proof, due to de Rham, uses Reidemeister torsion. Since the cellular chain complex of a lens space is acyclic when tensored with $\mathbb{Q}[x]$ for x a primitive k-th root of unity, one gets a based (by cells) acyclic complex $0 \to C_{2n-1} \to \cdots \to C_0 \to 0$, which gives us a well-defined nonzero determinant element in $\mathbb{Q}[x]$ (now called the associated element of K_1). This quantity is well-defined up to multiplication by a root of unity (and a sign). One now has to check that these actually determine the rotation numbers, a fact verified by Franz's lemma. See [**Mil66**] and [**Coh73**]. The second proof came much later and is due to Atiyah and Bott [**AB68**]. It uses index-theoretic ideas critically, and implies more about the topology of lens spaces. We will return to it a bit later.

After de Rham's theorem, it was very natural to ask, following Hurewicz, whether all homotopy equivalent simply-connected manifolds are diffeomorphic. (It was not until Milnor's examples of exotic spheres that mathematicians really considered seriously the existence of different categories of manifolds.) However, very classical results can be used to disprove this claim as well. Consider a sphere bundle over the sphere S^4 where the fiber is quite high-dimensional. Since $\pi_3(O(n)) = \mathbb{Z}$ for large n, we can construct an infinite number of these bundles by explicit clutching operations; their total spaces are distinguished by p_1. On the other hand, if we could nullhomotop the clutching maps in $\pi_3(\text{Isometries}(S^{n+1}))$ pushed into $\pi_3(\text{Selfmaps}(S^{n+1}))$, we would show that the total space is homotopy equivalent to a product. A little thought shows that $\pi_3(\text{Selfmaps}(S^{n+1}))$ is the same as the third stable homotopy group of spheres, which is finite by Serre's thesis. Combining this information, one quickly concludes that there are infinitely many manifolds homotopy equivalent to $S^4 \times S^{n+1}$ for large n, distinguished by p_1.

Much of our picture of high-dimensional manifolds comes from filtering the various strands arising in the above examples, analyzing them separately, and recombining them.

1.2. Before considering the parts that are most directly connected to operator K-theory, it is worthwhile to discuss the connection between the classification of manifolds and algebraic K-theory.

The aforementioned Reidemeister torsion invariant is an invariant of complexes defined under an acyclicity hypothesis. It is a computationally feasible shadow of a more basic invariant of homotopy equivalences, namely Whitehead torsion.

Let X and Y be finite complexes and $f : X \to Y$ a homotopy equivalence. Then using the chain complex of the mapping cylinder of f rel X or its universal cover, one obtains as before a finite-dimensional acyclic chain complex of based $\mathbb{Z}\pi$ chain complexes. The torsion $\tau(f)$ of f is the element of $K_1(\mathbb{Z}\pi)$ determined by means of the determinant, up to the indeterminacy of basis, which is a sign and element of π (viewed as a 1×1 matrix over the group ring). The quotient $K_1(\mathbb{Z}\pi)/\pm\pi$ is denoted by $\mathrm{Wh}(\pi)$.

A geometric interpretation of the vanishing of $\tau(f)$ is the following: say that X and Y are stably diffeomorphic (or, more naturally for this discussion, PL homeomorphic) if their regular neighborhoods in Euclidean space are diffeomorphic. The quantity $\tau(f)$ vanishes iff f is homotopic to a diffeomorphism between thickenings of X and Y. A homotopy equivalence with vanishing torsion is called a *simple homotopy equivalence*. As before, we recommend [**Coh73, Mil66**] for Whitehead's theory of simple homotopy and [**RS72**] for the theory of regular neighborhoods.

Remark. If we require X and Y to be manifolds, then one can ask that the stabilization only allow taking products with disks. Doing such does change the notion; the entire difference however is that we have discarded the topological K-theory. Two manifolds will be stably diffeomorphic in this restricted sense iff they have the same stable tangent bundle (in KO, or KPL for the PL analogue) and are simple homotopy equivalent. The proof of this fact is no harder than the polyhedral result.

1.3. Much deeper are unstable results. The prime example is Smale's h-cobordism theorem (or the Barden-Mazur-Stallings extension thereof).

Theorem 1. *Let M^n be a closed manifold of dimension at least 5; then $\{W^{n+1}$: M is one of two components of the boundary of W, and W deform retracts to both$\}$/diffeomorphism (or PL homeomorphism or homeomorphism) is in $1-1$ correspondence with Wh(π).*

The various W in the theorem are called h-cobordisms. The significance of this theorem should be obvious: it provides a way to produce diffeomorphisms from homotopy data. As such, it stands behind almost all of the high-dimensional classification theorems.