Introduction

These are the proceedings of the 2007 Clay Summer School on Homogeneous Flows, Moduli Spaces and Arithmetic which took place at the Centro di Ricerca Matematica Ennio De Giorgi in Pisa between June 11th to July 6th, 2007. More than 100 young researchers and graduate students attended this intensive four week school, as well as 18 lecturers and other established researchers.

As suggested by the name, the topic of this summer school consisted of two connected but distinct areas of active current research: flows on homogeneous spaces of algebraic groups (or Lie groups), and dynamics on moduli spaces of abelian or quadratic differentials on surfaces. These two subjects have common roots and have several important features in common; most importantly, they give concrete examples of dynamical systems with highly interesting behavior and a rich and powerful theory and moreover both have applications whose scope lies well outside that of the theory of dynamical systems.

The first three weeks of the summer school were devoted to the basic theory, and consisted mostly of three long lecture series. On the base of these lectures series, the following four sets of notes were written:

[1] *Interval exchange maps and translation surfaces* by J. C. Yoccoz
[2] *Unipotent flows and applications* by A. Eskin
[3] *Quantitative nondivergence and its diophantine applications* by D. Kleinbock
[4] *Diagonal actions on locally homogeneous spaces* by M. Einsiedler and E. Lindenstrauss

furthermore, there was a shorter lecture series

[5] *Fuchsian groups, geodesic flows on surfaces of constant negative curvature and symbolic coding of geodesics* by S. Katok.

Extensive notes for all the lecture series given in the first three weeks of the school are included in this proceedings volume (the content of the course by Eskin and Kleinbock has been separated into two different sets of notes). These papers were written to be read independently, and any of the five papers [1]–[5] could serve as a good starting point for the interested reader. More advanced topics were covered by several lecture series and individual lectures mostly given in the last week of the summer school; it was left to the discretion of the lecturers in these shorter courses whether to provide notes for these proceedings (though they were strongly encouraged to contribute). A list of these lecture notes with some additional details is given below.

The common root of both main topics of the summer school mentioned above lie (at least in part) in the theory of flows on surfaces of constant negative curvature, particularly the modular surface $\mathrm{SL}(2,\mathbb{Z})\backslash\mathbb{H}$ where pioneering work was done in the early 20th century by mathematicians such as Artin, Hedlund, Morse and others, and this theory has been developed much further in the times since. One highlight was the discovery that the geodesic flow on the modular surface is intimately connected to the continued fraction expansion of real numbers; indeed, when things are properly set up, one can view the continued fraction expansion as a symbolic coding of trajectories of the geodesic flow. These flows and their symbolic codings are carefully explained in Katok's notes; in later sections of that work, recent extensions of this classical result are also discussed.

One can view the modular surface $\mathrm{SL}(2,\mathbb{Z})\backslash\mathbb{H}$ in two ways: firstly, it can be viewed as the locally homogeneous space $\mathrm{SL}(2,\mathbb{Z})\backslash\mathrm{SL}(2,\mathbb{R})/\mathrm{SO}(2,\mathbb{R})$, in which case the geodesic flow as well as another important geometric flow — the horocycle flow — can be viewed as in the projection of trajectories of the one parameter groups

$$g_t = \begin{pmatrix} e^{t/2} & 0 \\ 0 & e^{-t/2} \end{pmatrix} \quad \text{and} \quad u_t = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \tag{1}$$

on the quotient space $\mathrm{SL}(2,\mathbb{Z})\backslash\mathrm{SL}(2,\mathbb{R})$. Another way to view $\mathrm{SL}(2,\mathbb{Z})\backslash\mathbb{H}$ is as a moduli space of flat structure (up to rotations) on a two dimensional torus. These two different points of view generalize to the two main themes of this Clay Summer School: flows on homogeneous spaces, and flows on moduli spaces of abelian or quadratic differentials (which are essentially fancy names for flat structures in two related but slightly different senses).

Flows on moduli spaces of flat structures. The torus is the only surface admitting a flat structure with no singularities. When one considers flat structures for surfaces of higher genus, one is forced to admit singularities: points where the total angles add up to more than 2π. It turns out that *interval exchange maps* play an important role in studying the analogue of the geodesic flow (sometimes called the Teichmüller geodesic flow) on these moduli spaces of flat structures. We recall that interval exchange maps are the following simple yet intriguing dynamical system: divide the unit interval $[0,1]$ into finitely many intervals $I_1, I_2, \ldots, I_d$ and then permute these intervals according to a permutation $\pi \in S_d$. Yoccoz' contribution to this proceedings provides an introduction to this theory, and provide full proofs of the most fundamental theorems (by Keane, Masur, Veech, Zorich) in the first ten sections and an introduction to some more advanced topics (Kontsevich-Zorich cocycle, cohomological equation, connected components of the moduli space, exponential mixing of the Teichmüller flow) in the last four sections. Further advanced topics are provided by notes based on the shorter lecture series

[6] *Chaoticity of the Teichmuller flow* by A. Avila

given in the last week of school; in these notes the interested reader can find surveys of the proof of two recent theorems: the simplicity of the Lyapunov spectrum for the Kontsevich-Zorich cocycle and that a typical interval exchange map with three or more intervals is weak mixing.

Flows on homogeneous spaces and applications to arithmetic. Flows on homogeneous spaces concerns the dynamics of group actions on quotient spaces $\Gamma\backslash G$ where G is usually taken to be either a **(i)** Lie group, **(ii)** an algebraic group over $\mathbb{R}$, **(iii)** an algebraic group over the p-adic numbers $\mathbb{Q}_p$, **(iv)** a product of algebraic groups as in (ii) and (iii) above, involving several different fields (sometimes called an S-algebraic group, where S refers to the set of "primes" p that are used[(1)].)

A simple case is the case of $G = \mathrm{SL}(2,\mathbb{R})$ and Γ a lattice in G, for instance $\Gamma = \mathrm{SL}(2,\mathbb{Z})$. In this case we have discussed (cf (1)) the action of two one-parameter subgroups of $\mathrm{SL}(2,\mathbb{R})$: the group g_t corresponding to the geodesic flow on the unit tangent bundle on $\Gamma\backslash\mathbb{H}$ and u_t which corresponds to the horocycle flow on the same space. These two flows behave very differently: the u_t-flow is very rigid, and one can algebraically classify orbit closures, invariant measures, measurable factors, self joinings, and even the asymptotic distribution of individual orbits. The g_t-flow is very flexible: it is certainly ergodic, but individual orbits can behave very badly and moreover is measure theoretically equivalent to a Bernoulli shift which has a wealth of measurable factors and self joinings.

The group u_t is an example of a *unipotent group.* In a fundamental series of papers published in 1990-91, M. Ratner proved that the above mentioned rigidity properties of u_t-flow are shared by all unipotent group actions on homogeneous spaces, in particular establishing in complete generality Raghunathan's conjecture about orbit closures for such actions (some cases of which were known previously, notably in the context of the Oppenheim conjecture discussed below). For the g_t-flow the situation is rather different: while a diagonalizable one parameter group in general behaves very much like g_t, higher dimensional diagonalizable groups seem to behave much more rigidly (though not as rigidly as unipotent group actions).

The notes by Eskin discuss in detail unipotent flows, with an emphasis on applications, in particularly regarding values attained by indefinite quadratic forms and Oppenheim's Conjecture. This long-standing conjecture was proved by Margulis in the mid-80s using homogeneous dynamics, and in particular unipotent dynamics. In dynamical terms, what Margulis has shown is that any bounded orbit of $\mathrm{SO}(2,1)$ on $\mathrm{SL}(3,\mathbb{Z})\backslash\mathrm{SL}(3,\mathbb{R})$ is closed. The notes also give a detailed exposition of a more delicate result giving precise asymptotics to the distribution of these values by Eskin, Margulis and Mozes (under certain assumptions on the signature of a quadratic form). Some of the ideas and methods used in the theory of unipotent flows, and in particular some of the ideas used by Ratner in her proof of the Measure Classification Theorem are also described in these notes. Eskin's notes contain also other interesting applications of unipotent rigidity as well as connections to dynamics of rational billiards.

Kleinbock's notes focus on a method originally introduced by Margulis to show that orbits of unipotent group actions do not diverge to infinity and developed significantly since. In particular, a quantitative version of the non-divergence statement due to S.G.Dani is an important ingredient in the proof of various versions of orbit closure and equidistribution theorems, including Ratner's Orbit Closure Theorem. However these techniques are more widely applicable and, in particular, were used by Kleinbock and Margulis to prove a conjecture of Sprindžuk on Diophantine approximations; this connection is also carefully discussed.

The notes by Einsiedler and Lindenstrauss discuss diagonalizable group actions, based mostly on work by the authors and by A. Katok in various combinations. A crucial role in current analysis of these actions is played by the concept of entropy.

[(1)]For this purpose ∞ is a prime and $\mathbb{Q}_\infty = \mathbb{R}$.

These notes give a detailed and self-contained account of the theory of the entropy in the locally homogeneous context, the construction of leafwise measures on foliations, and the connection between the two. Subsequently an account is given of two rather different and complementary methods to study measures invariant under multidimensional diagonalizability actions under suitable entropy assumptions, which go under the name of the *high entropy method* and the *low entropy method*. Two applications of this theory are also discussed: a partial result towards a conjecture of Littlewood on simultaneous Diophantine approximations, and how these techniques can be used to establish Arithmetic Quantum Unique Ergodicity on compact surfaces.

The material given in these three basic papers about homogeneous dynamics is complemented by the following two more advanced notes:

[7] *Counting and equidistribution on homogeneous spaces, via mixing and unipotent flows* by H. Oh

[8] *Equidistribution of Heegner points and L-functions* by G. Harcos

In Oh's notes, the use of equidistribution of unipotent flows (and the closely related but more quantitative mixing properties of diagonalizable flows) to count integer and rational points on certain varieties, a theme touched upon in Eskin's note, is developed further, and several state of the art applications are explained. The notes by Harcos gives some brief background in the theory of L-functions and how it relates to equidistribution of periodic orbits of the diagonal group in SL(2).

Semiclassical analysis and dynamics. One of the applications of the theory of diagonalizable actions discussed in the Einsiedler-Lindenstrauss note is establishing Arithmetic Quantum Unique Ergodicity for compact (arithmetic) surfaces. The Quantum Unique Ergodicity conjecture deals with the asymptotic distribution of eigenfunctions of the Laplacian; the arithmetic case is a very special case where the surface is arithmetic and eigenfunctions of the Laplacian are chosen so as to respect the rich set of symmetries of such surfaces. This question is considered from a completely different point of view in the notes

[9] *Eigenfunctions of the Laplacian on negatively curved manifolds: a semiclassical approach* by N. Anantharaman.

In these notes the basics of semiclassical analysis are reviewed, the connections between eigenvalues of the Laplacian and the geodesic flow, which have been discussed to some extent in the Einsiedler-Lindenstrauss notes, are developed in a much more systematic way, and very recent work relating entropy and limiting distributions of eigenfunctions of the Laplacian in general compact negatively curved manifolds (including the variable curvature case) is exposed.

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In addition to the authors of the notes listed above, the following mathematicians gave one or more lecture during this school: G. Forni, A. Gamburd, Y. Manin, G. Margulis, J. Marklof, M. Mirzakhani, S. Mozes, N. Templier, C. Ulcigrai, and A. Venkatesh. All lecturers and participants contributed to the enthusiastic and stimulating atmosphere at this school, and we thank them warmly for this.

For a variety of reasons, these lecture notes appear almost 3 years after the summer school. Quite a bit of work went into them, and indeed this is one of the reasons for the delay. They contain a substantial amount of material which can not be found in any textbook, and we hope you, the reader, would find them useful!

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