Preface

This volume is based on lectures given at the fourth Clay Mathematics Institute Summer School entitled "Harmonic Analysis, the Trace Formula, and Shimura Varieties." It was held at the Fields Institute in Toronto, Canada, from June 2 to June 27, 2003.

The main goal of the School was to introduce graduate students and young mathematicians to three broad and interrelated areas in the theory of automorphic forms. Much of the volume is comprised of the articles of Arthur, Kottwitz, and Milne. Although these articles are based on lectures given at the school, the authors have chosen to go well beyond what was discussed there, in order to provide both a sense of the underlying structure of the subject and a working knowledge of some of its techniques. They were written to be self-contained in some places, and to be used in conjunction with given references in others. We hope the volume will convey the depth and beauty of this challenging field, in which there yet remains so much to be discovered—perhaps some of it by you, the reader!

The theory of automorphic forms is formulated in terms of reductive algebraic groups. This is sometimes a serious obstacle for mathematicians whose background does not include Lie groups and Lie algebras. The monograph is by no means intended to exclude such mathematicians, even though the theory of reductive groups was an informal prerequisite for the Summer School. Some modest familiarity with the language of algebraic groups is often sufficient, at least to get started. For this reason, we have generally resisted the temptation to work with specific matrix groups. The short article of Murnaghan contains a summary of some of the basic properties of reductive algebraic groups that are used elsewhere in the monograph.

Much of the modern theory of automorphic forms is governed by two fundamental problems that are at the heart of the Langlands program. One is Langlands' principle of functoriality. The other is the general analogue of the Shimura-Taniyama-Weil conjecture on modular elliptic curves. (See [**A**] and [**L**, §2].) These problems are among the deepest questions in mathematics. It is premature to try to guess what various techniques will play a role in their ultimate resolution. However, the trace formula and the theory of Shimura varieties are both likely to be an essential part of the story. They have already been used to establish significant special cases.

The trace formula has perhaps been more closely identified with the first problem. Special cases of functoriality arise naturally from the conjectural theory of endoscopy, in which a comparison of trace formulas would be used to characterize the internal structure of the automorphic representations of a given group. (See [**Sh**] for a discussion of the first case to be investigated.) Likewise, Shimura varieties are usually associated with the second problem. As higher dimensional analogues of modular curves, they are attached by definition to certain reductive groups. In many cases, it has been possible to establish reciprocity laws between ℓ-adic Galois representations on their cohomology groups and automorphic representations of the corresponding reductive groups. These laws can be formulated as an explicit formula for the zeta function of a Shimura variety in terms of automorphic L-functions. (See [**K**] for a discussion of the rough form such a formula is expected to take. The word "rough" should be taken seriously, given the current limitations of our understanding.)

The work of Wiles that led to a proof of Fermat's Last Theorem suggests that the two problems are inextricably linked. This is already apparent in the reciprocity laws that have been established for Shimura varieties. Indeed, the conjectural formula for the zeta function of a general Shimura variety requires the theory of endoscopy even to state. Moreover, the proof of these reciprocity laws requires a comparison of the (automorphic) trace formula with an (ℓ-adic) Lefschetz trace formula. Some of the most striking parts of the argument are in the comparison of the various terms in the two formulas. The most sophisticated Shimura varieties for which there are complete results are the so-called Picard modular surfaces. (See [**LR**], especially the summary on pp. 255–302.) Picard modular surfaces are attached to unitary groups in three variables. It is no coincidence that the theory of endoscopy has also been established for these groups, thereby yielding a classification of their automorphic representations [**R**].

There is some discussion of these problems in the articles of Arthur and Milne. However, the articles of both Arthur and Milne really are intended as introductions, despite their length. The theory of endoscopy, and the automorphic description of zeta functions of Shimura varieties, are at the forefront of present day research. They are for the most part beyond the scope of this monograph.

The local terms in the trace formula are essentially analytic objects. They include the invariant orbital integrals and irreducible characters that are the basis for Harish-Chandra's theory of local harmonic analysis. They also include weighted orbital integrals and weighted characters, objects that arose for the first time with the trace formula. The article of Kottwitz is devoted to the general study of these terms at p-adic places. It is a largely self-contained course, which covers many of Harish-Chandra's basic results in invariant harmonic analysis, as well as their weighted, noninvariant analogues.

The article of DeBacker focuses on the phenomenon of homogeneity in invariant harmonic analysis at p-adic places. It concerns quantitative forms of some of the basic theorems of p-adic harmonic analysis, such as Howe's finiteness theorem and Harish-Chandra's local character expansion. The article also explains how homogeneity enters into Waldspurger's analysis of stability for linear combinations of nilpotent orbital integrals.

There are subtle questions concerning the terms in the trace formula that go beyond those treated by Kottwitz and DeBacker. The most basic of these is known as the fundamental lemma, even though it is still largely conjectural.[1] The article by Hales contains a precise statement of the conjecture and some remarks on progress toward a general proof. The fundamental lemma occupies a unique place in the theory. It is a critical ingredient in the comparison of trace formulas that is part of the theory of endoscopy. It has an equally indispensable role in the comparison of (automorphic and ℓ-adic) trace formulas needed to establish reciprocity laws for Shimura varieties.

Some Shimura varieties are projective, which is to say that they are compact as complex varieties. They correspond to reductive groups over $\mathbb{Q}$ that are anisotropic. The trace formula in this case simplifies considerably. It reduces to the Selberg trace formula for compact quotient. On the other hand, the arithmetic geometry of such varieties is still very rich. In particular, the comparison of individual terms in the

[1]Moreover, the term *lemma* is ultimately a gross understatement.

two kinds of trace formulas is of major interest. There is a great deal left to be done, but it is in this case that there has been the most progress.

If the Shimura variety is not projective, the comparison is more sophisticated. It has to be based on the relationship between L^2-cohomology and intersection cohomology, conjectured by Zucker, and established by Saper and Stern, and Looijenga. The article of Goresky describes several compactifications of open Shimura varieties and their relations with associated cohomology groups. Goresky's article also serves as an introduction to work of Goresky and MacPherson, in which weighted cohomology complexes on the reductive Borel-Serre compactification are used to obtain a Lefschetz formula for the intersection cohomology of the Baily-Borel compactification. According to Zucker's conjecture, this last formula is equivalent to the relevant form of the automorphic trace formula. There remains the important open problem of establishing a corresponding ℓ-adic Lefschetz formula that can be compared with either one of these two formulas.

The reciprocity laws proved for Picard modular surfaces in [**LR**] apply to places of good reduction. The same restriction has been implicit in our discussion of other Shimura varieties. In the final analysis, one would like to establish reciprocity laws between ℓ-adic Galois representations and automorphic representations that apply to all places. The theory of Shimura varieties at places of bad reduction is considerably less developed, although there has certainly been progress. The article of Haines is a survey of recent work in this direction, concentrating on the case of level structures of parahoric type. It also touches upon the problem of comparing the automorphic trace formula with the Lefschetz formula, now in the context of bad reduction.

The article of Sarnak concerns the classical Ramanujan conjecture for modular forms and its higher dimensional analogues. Langlands has shown that the generalized Ramanujan conjecture is a consequence of the principle of functoriality. Conversely, it is possible that the generalized Ramanujan conjecture could play a critical role in the study of those cases of functoriality that are not part of the theory of endoscopy. Sarnak describes the present state of the conjecture and discusses various techniques that have been successfully applied to special cases.

We have tried to present the contents of the monograph from a unified perspective. Our description has been centered around two fundamental problems that are the essential expression of the Langlands program. The two problems ought to be treated as signposts, which give direction to current work, but which point to destinations that will not be reached in the foreseeable future. The reader is free to draw whatever inspiration from them his or her temperament permits. In any case, many of the questions discussed in the various articles here are of great interest in their own right. In point of fact, there is probably too much in the monograph for anyone to learn in a limited period of time. Perhaps the best strategy for a beginner would be to start with one or two articles of special interest, and try to master them.

As we have mentioned, participants were encouraged to bring a prior understanding of the basic properties of algebraic groups. The theory of reductive groups is rooted in the structure of complex semisimple Lie algebras, for which [**Se**] and [**H**] are good references. As for algebraic groups themselves, a familiarity with many of the topics in [**B**] or [**Sp**] is certainly desirable, though perhaps not essential.

Participants were also assumed to have some knowledge of number theory. The main theorem of class field theory is reviewed without proof in the article of Milne. A complete treatment can be found in [**CF**]. Tate's article on global class field theory in this reference contains a particularly good introduction to the theory. The thesis of Tate, reprinted as a separate article in [**CF**], is also recommended for its introduction to adeles and its construction of the basic abelian automorphic L-functions.

A reader might also want to consult other general articles in automorphic forms. A good introductory reference to the general theory of automorphic forms is the proceedings of the Edinburgh instructional conference [**BK**].

This Clay Mathematics Institute Summer School could not have taken place without the efforts of many people. We deeply appreciate the role of the Clay Mathematics Institute in making this summer school possible, and thank Vida Salahi in particular for the care and attention she exercised in bringing the volume to its final form. We are most grateful to the staff of the Fields Institute, who did such a superb job of making the School run smoothly. We are equally indebted to all the lecturers, not only for agreeing to take part in the School, but also for providing the texts collected in this volume. Last, but surely not least, we would like to thank the participants, whose enthusiastic response made it all worthwhile.

James Arthur, David Ellwood, Robert Kottwitz.
August, 2005.

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Summer School Lecture Courses: June 2–20, 2003

Introduction to the trace formula
J. Arthur, June 2–20

Introduction to Shimura varieties
J. Milne, June 2–20

Background from algebraic groups
F. Murnaghan, June 2–6

Harmonic analysis on reductive groups and Lie algebras
R. Kottwitz, June 9–20

Advanced Short Courses: June 23–27, 2003

An introduction to homogeneity with applications
S. DeBacker

Geometry and topology of compactifications of modular varieties
M. Goresky

Bad reduction of Shimura varieties
T. Haines

An introduction to the fundamental lemma
T. Hales

Analytic aspects of automorphic forms
P. Sarnak