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The path to recent progress on small gaps between primes

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Abstract. We present the development of ideas which led to our recent findings about the existence of small gaps between primes.

1. Introduction

In the articles *Primes in Tuples I & II* ([**GPYa**], [**GPYb**]) we have presented the proofs of some assertions about the existence of small gaps between prime numbers which go beyond the hitherto established results. Our method depends on tuple approximations. However, the approximations and the way of applying the approximations has changed over time, and some comments in this paper may provide insight as to the development of our work.

First, here is a short narration of our results. Let

$$\theta(n) := \begin{cases} \log n & \text{if } n \text{ is prime,} \\ 0 & \text{otherwise,} \end{cases} \tag{1}$$

and

$$\Theta(N; q, a) := \sum_{\substack{n \le N \\ n \equiv a \ (\bmod\ q)}} \theta(n). \tag{2}$$

In this paper N will always be a large integer, p will denote a prime number, and p_n will denote the n-th prime. The prime number theorem says that

$$\lim_{x\to\infty} \frac{|\{p : \ p \le x\}|}{\frac{x}{\log x}} = 1, \tag{3}$$

and this can also be expressed as

$$\sum_{n \le x} \theta(n) \sim x \qquad \text{as } x \to \infty. \tag{4}$$

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It follows trivially from the prime number theorem that

$$\liminf_{n\to\infty} \frac{p_{n+1}-p_n}{\log p_n} \le 1. \tag{5}$$

By combining former methods with a construction of certain (rather sparsely distributed) intervals which contain more primes than the expected number by a factor of e^γ, Maier [**Mai88**] had reached the best known result in this direction that

$$\liminf_{n\to\infty} \frac{p_{n+1}-p_n}{\log p_n} \le 0.24846... \ . \tag{6}$$

It is natural to expect that modulo q the primes would be almost equally distributed among the reduced residue classes. The deepest knowledge on primes which plays a role in our method concerns a measure of the distribution of primes in reduced residue classes referred to as the level of distribution of primes in arithmetic progressions. We say that the primes have *level of distribution* α if

$$\sum_{q\le Q} \max_{\substack{a\\(a,q)=1}} \left|\Theta(N;q,a) - \frac{N}{\phi(q)}\right| \ll \frac{N}{(\log N)^A} \tag{7}$$

holds for any $A > 0$ and any arbitrarily small fixed $\epsilon > 0$ with

$$Q = N^{\alpha-\epsilon}. \tag{8}$$

The *Bombieri-Vinogradov theorem* provides the level $\frac{1}{2}$, while the *Elliott-Halberstam conjecture* asserts that the primes have level of distribution 1.

The Bombieri-Vinogradov theorem allows taking $Q = N^{\frac{1}{2}}(\log N)^{-B(A)}$ in (7), by virtue of which we have proved unconditionally in [**GPYa**] that for any fixed $r \ge 1$,

$$\liminf_{n\to\infty} \frac{p_{n+r}-p_n}{\log p_n} \le (\sqrt{r}-1)^2 \ ; \tag{9}$$

in particular,

$$\liminf_{n\to\infty} \frac{p_{n+1}-p_n}{\log p_n} = 0. \tag{10}$$

In fact, assuming that the level of distribution of primes is α, we obtain more generally than (9) that, for $r \ge 2$,

$$\liminf_{n\to\infty} \frac{p_{n+r}-p_n}{\log p_n} \le (\sqrt{r}-\sqrt{2\alpha})^2. \tag{11}$$

Furthermore, assuming that $\alpha > \frac{1}{2}$, there exists an explicitly calculable constant $C(\alpha)$ such that for $k \ge C(\alpha)$ any sequence of k-tuples

$$\{(n+h_1, n+h_2, \dots, n+h_k)\}_{n=1}^{\infty}, \tag{12}$$

with the set of distinct integers $\mathcal{H} = \{h_1, h_2, \dots, h_k\}$ *admissible* in the sense that $\prod_{i=1}^{k}(n+h_i)$ has no fixed prime factor for every n, contains at least two primes infinitely often. For instance if $\alpha \ge 0.971$, then this holds for $k \ge 6$, giving

$$\liminf_{n\to\infty}(p_{n+1}-p_n) \le 16, \tag{13}$$

in view of the shortest admissible 6-tuple $(n, n+4, n+6, n+10, n+12, n+16)$.

By incorporating Maier's method into ours in [**GPY06**] we improved (9) to

$$\liminf_{n\to\infty} \frac{p_{n+r}-p_n}{\log p_n} \le e^{-\gamma}(\sqrt{r}-1)^2, \tag{14}$$

along with an extension for primes in arithmetic progressions where the modulus can tend slowly to infinity as a function of p_n.

In [**GPYb**] the result (10) was considerably improved to

$$\liminf_{n\to\infty} \frac{p_{n+1}-p_n}{(\log p_n)^{\frac{1}{2}}(\log\log p_n)^2} < \infty. \tag{15}$$

In fact, the methods of [**GPYb**] lead to a much more general result: When $\mathcal{A} \subseteq \mathbb{N}$ is a sequence satisfying $\mathcal{A}(N) := |\{n; n \le N, n \in \mathcal{A}\}| > C(\log N)^{1/2}(\log\log N)^2$ for all sufficiently large N, infinitely many of the differences of two elements of $\mathcal{A}$ can be expressed as the difference of two primes.

2. Former approximations by truncated divisor sums

The von Mangoldt function

$$\Lambda(n) := \begin{cases} \log p & \text{if} \quad n = p^m,\ m \in \mathbb{Z}^+, \\ 0 & \text{otherwise}, \end{cases} \tag{16}$$

can be expressed as

$$\Lambda(n) = \sum_{d|n} \mu(d)\log(\frac{R}{d}) \qquad \text{for } n > 1. \tag{17}$$

Since the proper prime powers contribute negligibly, the prime number theorem (4) can be rewritten as

$$\psi(x) := \sum_{n\le x} \Lambda(n) \sim x \qquad \text{as } x \to \infty. \tag{18}$$

It is natural to expect that the truncated sum

$$\Lambda_R(n) := \sum_{\substack{d|n \\ d\le R}} \mu(d)\log(\frac{R}{d}) \qquad \text{for } n \ge 1. \tag{19}$$

mimics the behaviour of $\Lambda(n)$ on some averages.

The beginning of our line of research is Goldston's [**G92**] alternative rendering of the proof of Bombieri and Davenport's theorem on small gaps between primes. Goldston replaced the application of the circle method in the original proof by the use of the truncated divisor sum (19). The use of functions like $\Lambda_R(n)$ goes back to Selberg's work [**Sel42**] on the zeros of the Riemann zeta-function $\zeta(s)$. The most beneficial feature of the truncated divisor sums is that they can be used in place of $\Lambda(n)$ on some occasions when it is not known how to work with $\Lambda(n)$ itself. The principal such situation arises in counting the primes in tuples. Let

$$\mathcal{H} = \{h_1, h_2, \ldots, h_k\} \ \text{ with } \ 1 \le h_1, \ldots, h_k \le h \ \text{ distinct integers} \tag{20}$$

(the restriction of h_i to positive integers is inessential; the whole set $\mathcal{H}$ can be shifted by a fixed integer with no effect on our procedure), and for a prime p denote by $\nu_p(\mathcal{H})$ the number of distinct residue classes modulo p occupied by the elements of $\mathcal{H}$. The singular series associated with the k-tuple $\mathcal{H}$ is defined as

$$\mathfrak{S}(\mathcal{H}) := \prod_p (1 - \frac{1}{p})^{-k}(1 - \frac{\nu_p(\mathcal{H})}{p}). \tag{21}$$

Since $\nu_p(\mathcal{H}) = k$ for $p > h$, the product is convergent. The admissibility of $\mathcal{H}$ is equivalent to $\mathfrak{S}(\mathcal{H}) \neq 0$, and to $\nu_p(\mathcal{H}) \neq p$ for all primes. Hardy and Littlewood [**HL23**] conjectured that

$$\sum_{n \le N} \Lambda(n;\mathcal{H}) := \sum_{n \le N} \Lambda(n+h_1)\cdots\Lambda(n+h_k) = N(\mathfrak{S}(\mathcal{H}) + o(1)), \quad \text{as } N \to \infty. \tag{22}$$

The prime number theorem is the $k = 1$ case, and for $k \ge 2$ the conjecture remains unproved. (This conjecture is trivially true if $\mathcal{H}$ is inadmissible).

A simplified version of Goldston's argument in [**G92**] was given in [**GY03**] as follows. To obtain information on small gaps between primes, let

$$\psi(n,h) := \psi(n+h) - \psi(n) = \sum_{n < m \le n+h} \Lambda(m), \quad \psi_R(n,h) := \sum_{n < m \le n+h} \Lambda_R(m), \tag{23}$$

and consider the inequality

$$\sum_{N < n \le 2N} (\psi(n,h) - \psi_R(n,h))^2 \ge 0. \tag{24}$$

The strength of this inequality depends on how well $\Lambda_R(n)$ approximates $\Lambda(n)$. On multiplying out the terms and using from [**G92**] the formulas

(25)
$$\sum_{n \le N} \Lambda_R(n)\Lambda_R(n+k) \sim \mathfrak{S}(\{0,k\})N, \quad \sum_{n \le N} \Lambda(n)\Lambda_R(n+k) \sim \mathfrak{S}(\{0,k\})N \quad (k \neq 0)$$

(26)
$$\sum_{n \le N} \Lambda_R(n)^2 \sim N \log R, \quad \sum_{n \le N} \Lambda(n)\Lambda_R(n) \sim N \log R,$$

valid for $|k| \le R \le N^{\frac{1}{2}}(\log N)^{-A}$, gives, taking $h = \lambda \log N$ with $\lambda \ll 1$,

(27)
$$\sum_{N < n \le 2N} (\psi(n+h) - \psi(n))^2 \ge (hN \log R + Nh^2)(1 - o(1)) \ge (\frac{\lambda}{2} + \lambda^2 - \epsilon)N(\log N)^2$$

(in obtaining this one needs the two-tuple case of Gallagher's singular series average given in (46) below, which can be traced back to Hardy and Littlewood's and Bombieri and Davenport's work). If the interval $(n, n+h]$ never contains more than one prime, then the left-hand side of (27) is at most

$$\log N \sum_{N < n \le 2N} (\psi(n+h) - \psi(n)) \sim \lambda N(\log N)^2, \tag{28}$$

which contradicts (27) if $\lambda > \frac{1}{2}$, and thus one obtains

$$\liminf_{n \to \infty} \frac{p_{n+1} - p_n}{\log p_n} \le \frac{1}{2}. \tag{29}$$

Later on Goldston et al. in [**FG96**], [**FG99**], [**G95**], [**GY98**], [**GY01**], [**GYa**] applied this lower-bound method to various problems concerning the distribution