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The density of integral solutions for pairs of diagonal cubic equations

Jörg Brüdern and Trevor D. Wooley

Abstract. We investigate the number of integral solutions possessed by a pair of diagonal cubic equations in a large box. Provided that the number of variables in the system is at least thirteen, and in addition the number of variables in any non-trivial linear combination of the underlying forms is at least seven, we obtain a lower bound for the order of magnitude of the number of integral solutions consistent with the product of local densities associated with the system.

1. Introduction

This paper is concerned with the solubility in integers of the equations

$$a_1x_1^3 + a_2x_2^3 + \ldots + a_sx_s^3 = b_1x_1^3 + b_2x_2^3 + \ldots + b_sx_s^3 = 0, \tag{1.1}$$

where $(a_i, b_i) \in \mathbb{Z}^2 \setminus \{\mathbf{0}\}$ are fixed coefficients. It is natural to enquire to what extent the Hasse principle holds for such systems of equations. Cook [**C85**], refining earlier work of Davenport and Lewis [**DL66**], has analysed the local solubility problem with great care. He showed that when $s \geq 13$ and p is a prime number with $p \neq 7$, then the system (1.1) necessarily possesses a non-trivial solution in $\mathbb{Q}_p$. Here, by *non-trivial solution*, we mean any solution that differs from the obvious one in which $x_j = 0$ for $1 \leq j \leq s$. No such conclusion can be valid for $s \leq 12$, for there may then be local obstructions for any given set of primes p with $p \equiv 1 \pmod 3$; see [**BW06**] for an example that illuminates this observation. The 7-adic case, moreover, is decidedly different. For $s \leq 15$ there may be 7-adic obstructions to the solubility of the system (1.1), and so it is only when $s \geq 16$ that the existence of non-trivial solutions in $\mathbb{Q}_7$ is assured. This much was known to Davenport and Lewis [**DL66**].

Were the Hasse principle to hold for systems of the shape (1.1), then in view of the above discussion concerning the local solubility problem, the existence of

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integer solutions to the equations (1.1) would be decided in $\mathbb{Q}_7$ alone whenever $s \geq 13$. Under the more stringent hypothesis $s \geq 14$, this was confirmed by the first author [**B90**], building upon the efforts of Davenport and Lewis [**DL66**], Cook [**C72**], Vaughan [**V77**] and Baker and Brüdern [**BB88**] spanning an interval of more than twenty years. In a recent collaboration [**BW06**] we have been able to add the elusive case $s = 13$, and may therefore enunciate the following conclusion.

THEOREM 1. *Suppose that $s \geq 13$. Then for any choice of coefficients $(a_j, b_j) \in \mathbb{Z}^2 \setminus \{\mathbf{0}\}$ $(1 \leq j \leq s)$, the simultaneous equations* (1.1) *possess a non-trivial solution in rational integers if and only if they admit a non-trivial solution in $\mathbb{Q}_7$.*

Now let $\mathcal{N}_s(P)$ denote the number of solutions of the system (1.1) in rational integers $x_1, \ldots, x_s$ satisfying the condition $|x_j| \leq P$ $(1 \leq j \leq s)$. When s is large, a naïve application of the philosophy underlying the circle method suggests that $\mathcal{N}_s(P)$ should be of order P^{s-6} in size, but in certain cases this may be false even in the absence of local obstructions. This phenomenon is explained by the failure of the Hasse principle for certain diagonal cubic forms in four variables. When $s \geq 10$ and $b_1, \ldots, b_s \in \mathbb{Z} \setminus \{0\}$, for example, the simultaneous equations

$$5x_1^3 + 9x_2^3 + 10x_3^3 + 12x_4^3 = b_1x_1^3 + b_2x_2^3 + \ldots + b_sx_s^3 = 0 \tag{1.2}$$

have non-trivial (and non-singular) solutions in every p-adic field $\mathbb{Q}_p$ as well as in $\mathbb{R}$, yet all solutions in rational integers must satisfy the condition $x_i = 0$ $(1 \leq i \leq 4)$. The latter must hold, in fact, independently of the number of variables. For such examples, therefore, one has $\mathcal{N}_s(P) = o(P^{s-6})$ when $s \geq 9$, whilst for $s \geq 12$ one may show that $\mathcal{N}_s(P)$ is of order P^{s-7}. For more details, we refer the reader to the discussion surrounding equation (1.2) of [**BW06**]. This example also shows that weak approximation may fail for the system (1.1), even when s is large.

In order to measure the extent to which a system (1.1) may resemble the pathological example (1.2), we introduce the number q_0, which we define by

$$q_0 = \min_{(c,d) \in \mathbb{Z}^2 \setminus \{\mathbf{0}\}} \operatorname{card}\{1 \leq j \leq s : ca_j + db_j \neq 0\}.$$

This important invariant of the system (1.1) has the property that as q_0 becomes larger, the counting function $\mathcal{N}_s(P)$ behaves more tamely. Note that in the example (1.2) discussed above one has $q_0 = 4$ whenever $s \geq 8$.

THEOREM 2. *Suppose that $s \geq 13$, and that $(a_j, b_j) \in \mathbb{Z}^2 \setminus \{\mathbf{0}\}$ $(1 \leq j \leq s)$ satisfy the condition that the system* (1.1) *admits a non-trivial solution in $\mathbb{Q}_7$. Then whenever $q_0 \geq 7$, one has $\mathcal{N}_s(P) \gg P^{s-6}$.*

The conclusion of Theorem 2 was obtained in our recent paper [**BW06**] for all cases wherein $q_0 \geq s - 5$. This much suffices to establish Theorem 1; see §8 of [**BW06**] for an account of this deduction. Our main objective in this paper is a detailed discussion of the cases with $7 \leq q_0 \leq s - 6$. We remark that the arguments of this paper as well as those in [**BW06**] extend to establish weak approximation for the system (1.1) when $s \geq 13$ and $q_0 \geq 7$. In the special cases in which $s = 13$ and q_0 is equal to either 5 or 6, a conditional proof of weak approximation is possible by invoking recent work of Swinnerton-Dyer [**SD01**], subject to the as yet unproven finiteness of the Tate-Shafarevich group for elliptic curves over quadratic fields. Indeed, equipped with the latter conclusion, for these particular values of q_0 one may relax the condition on s beyond that addressed by Theorem 2. When $s = 13$

and $q_0 \leq 4$, on the other hand, weak approximation fails in general, as we have already seen in the discussion accompanying the system (1.2).

The critical input into the proof of Theorem 2 is a certain arithmetic variant of Bessel's inequality established in [**BW06**]. We begin in §2 by briefly sketching the principal ideas underlying this innovation. In §3 we prepare the ground for an application of the Hardy-Littlewood method, deriving a lower bound for the major arc contribution in the problem at hand. Some delicate footwork in §4 establishes a mean value estimate that, in all circumstances save for particularly pathological situations, leads in §5 to a viable complementary minor arc estimate sufficient to establish Theorem 2. The latter elusive situations are handled in §6 via an argument motivated by our recent collaboration [**BKW01a**] with Kawada, and thereby we complete the proof of Theorem 2. Finally, in §7, we make some remarks concerning the extent to which our methods are applicable to systems containing fewer than 13 variables.

Throughout, the letter ε will denote a sufficiently small positive number. We use $\ll$ and $\gg$ to denote Vinogradov's well-known notation, implicit constants depending at most on ε, unless otherwise indicated. In an effort to simplify our analysis, we adopt the convention that whenever ε appears in a statement, then we are implicitly asserting that for each $\varepsilon > 0$ the statement holds for sufficiently large values of the main parameter. Note that the "value" of ε may consequently change from statement to statement, and hence also the dependence of implicit constants on ε. Finally, from time to time we make use of vector notation in order to save space. Thus, for example, we may abbreviate $(c_1, \dots, c_t)$ to $\mathbf{c}$.

2. An arithmetic variant of Bessel's inequality

The major innovation in our earlier paper [**BW06**] is an arithmetic variant of Bessel's inequality that sometimes provides good mean square estimates for Fourier coefficients averaged over sparse sequences. Since this tool plays a crucial role also in our current excursion, we briefly sketch the principal ideas. When P and R are real numbers with $1 \leq R \leq P$, we define the set of smooth numbers $\mathcal{A}(P,R)$ by

$$\mathcal{A}(P,R) = \{n \in \mathbb{N} \cap [1,P] \,:\, p \text{ prime and } p|n \Rightarrow p \leq R\}.$$

The Fourier coefficients that are to be averaged arise in connection with the smooth cubic Weyl sum $h(\alpha) = h(\alpha; P, R)$, defined by

$$h(\alpha; P, R) = \sum_{x \in \mathcal{A}(P,R)} e(\alpha x^3), \tag{2.1}$$

where here and later we write $e(z)$ for $\exp(2\pi i z)$. The sixth moment of this sum has played an important role in many applications in recent years, and that at hand is no exception to the rule. Write $\xi = (\sqrt{2833} - 43)/41$. Then as a consequence of the work of the second author [**W00**], given any positive number ε, there exists a positive number $\eta = \eta(\varepsilon)$ with the property that whenever $1 \leq R \leq P^\eta$, one has

$$\int_0^1 |h(\alpha; P, R)|^6 \, d\alpha \ll P^{3+\xi+\varepsilon}. \tag{2.2}$$

We assume henceforth that whenever R appears in a statement, either implicitly or explicitly, then $1 \leq R \leq P^\eta$ with η a positive number sufficiently small in the context of the upper bound (2.2).

The Fourier coefficients over which we intend to average are now defined by

$$\psi(n)=\int_0^1 |h(\alpha)|^5 e(-n\alpha)\,d\alpha. \tag{2.3}$$

An application of Parseval's identity in combination with conventional circle method technology readily shows that $\sum_n \psi(n)^2$ is of order P^7. Rather than average $\psi(n)$ in mean square over all integers, we instead restrict to the sparse sequence consisting of differences of two cubes, and establish the bound

$$\sum_{1\le x,y\le P}\psi(x^3-y^3)^2 \ll P^{6+\xi+4\varepsilon}. \tag{2.4}$$

Certain contributions to the sum on the left hand side of (2.4) are easily estimated. By Hua's Lemma (see Lemma 2.5 of [**V97**]) and a consideration of the underlying Diophantine equations, one has

$$\int_0^1 |h(\alpha)|^4\,d\alpha \ll P^{2+\varepsilon}.$$

On applying Schwarz's inequality to (2.3), we therefore deduce from (2.2) that the estimate $\psi(n)=O(P^{5/2+\xi/2+\varepsilon})$ holds uniformly in n. We apply this upper bound with $n=0$ in order to show that the terms with $x=y$ contribute at most $O(P^{6+\xi+2\varepsilon})$ to the left hand side of (2.4). The integers x and y with $1\le x,y\le P$ and $|\psi(x^3-y^3)|\le P^{2+\xi/2+2\varepsilon}$ likewise contribute at most $O(P^{6+\xi+4\varepsilon})$ within the summation of (2.4). We estimate the contribution of the remaining Fourier coefficients by dividing into dyadic intervals. When T is a real number with

$$P^{2+\xi/2+2\varepsilon}\le T\le P^{5/2+\xi/2+2\varepsilon}, \tag{2.5}$$

define $\mathcal{Z}(T)$ to be the set of ordered pairs $(x,y)\in\mathbb{N}^2$ with

$$1\le x,y\le P,\quad x\ne y\quad\text{and}\quad T\le|\psi(x^3-y^3)|\le 2T, \tag{2.6}$$

and write $Z(T)$ for $\operatorname{card}(\mathcal{Z}(T))$. Then on incorporating in addition the contributions of those terms already estimated, a familiar dissection argument now demonstrates that there is a number T satisfying (2.5) for which

$$\sum_{1\le x,y\le P}\psi(x^3-y^3)^2 \ll P^{6+\xi+4\varepsilon}+P^{\varepsilon}T^2Z(T). \tag{2.7}$$

An upper bound for $Z(T)$ at this point being all that is required to complete the proof of the estimate (2.4), we set up a mechanism for deriving such an upper bound that has its origins in work of Brüdern, Kawada and Wooley [**BKW01a**] and Wooley [**W02**]. Let $\sigma(n)$ denote the sign of the real number $\psi(n)$ defined in (2.3), with the convention that $\sigma(n)=0$ when $\psi(n)=0$, so that $\psi(n)=\sigma(n)|\psi(n)|$. Then on forming the exponential sum

$$K_T(\alpha)=\sum_{(x,y)\in\mathcal{Z}(T)}\sigma(x^3-y^3)e(\alpha(y^3-x^3)),$$

we find from (2.3) and (2.6) that

$$\int_0^1 |h(\alpha)|^5K_T(\alpha)\,d\alpha\ge TZ(T).$$