

Heegner points and non-vanishing of Rankin/Selberg L -functions

Philippe Michel and Akshay Venkatesh

ABSTRACT. We discuss the nonvanishing of central values $L(\frac{1}{2}, f \otimes \chi)$, where f is a fixed automorphic form on $GL(2)$ and χ varies through class group characters of an imaginary quadratic field $K = \mathbb{Q}(\sqrt{-D})$, as D varies; we prove results of the nature that at least $D^{1/5000}$ such twists are nonvanishing. We also discuss the related question of the rank of a fixed elliptic curve E/\mathbb{Q} over the Hilbert class field of $\mathbb{Q}(\sqrt{-D})$, as D varies. The tools used are results about the distribution of Heegner points, as well as subconvexity bounds for L -functions.

1. Introduction

The problem of studying the non-vanishing of central values of automorphic L -functions arise naturally in several contexts ranging from analytic number theory, quantum chaos and arithmetic geometry and can be approached by a great variety of methods (ie. via analytic, geometric spectral and ergodic techniques or even a blend of them).

Amongst the many interesting families that may occur, arguably one of the most attractive is the family of (the central values of) *twists by class group characters*: Let f be a modular form on $PGL(2)$ over \mathbb{Q} and K a quadratic field of discriminant D . If χ is a ring class character associated to K , we may form the L -function $L(s, f \otimes \chi)$: the Rankin-Selberg convolution of f with the θ -series $g_\chi(z) = \sum_{\{0\} \neq \mathfrak{a} \subset \mathcal{O}_K} \chi(\mathfrak{a}) e(N(\mathfrak{a})z)$. Here g_χ is a holomorphic Hecke-eigenform of weight 1 on $\Gamma_0(D)$ with Nebentypus χ_K and a cusp form iff χ is not a quadratic character¹.

2000 *Mathematics Subject Classification*. Primary 11F66, Secondary 11F67, 11M41.

Key words and phrases. Automorphic L -functions, Central Values, Subconvexity, Equidistribution.

The research of the first author is partially supported by the RMT-network ‘‘Arithmetic Algebraic Geometry’’ and by the ‘‘RAP’’ network of the R egion Languedoc-Roussillon.

The second author was supported by the Clay Mathematics Institute and also acknowledges support through NSF grants 02045606 and NSF Grants DMS0111298. He thanks the Institute of Advanced Study for providing excellent working conditions.

¹Equivalently, one can define $L(s, f \otimes \chi)$ as $L(s, \Pi_f \otimes \chi)$, where Π_f is the base-change to K of the automorphic representation underlying f , and χ is regarded as a character of $\mathbb{A}_K^\times/K^\times$.

We will always assume that the conductor of f is coprime to the discriminant of K . In that case the sign of the functional equation equals $\pm \left(\frac{-D}{N}\right)$, where one takes the $+$ sign in the case when f is Maass, and the $-$ sign if f is weight 2 holomorphic (these are the only cases that we shall consider).

Many lovely results have been proved in this context: we refer the reader to §1.3 for a review of some of these results. A common theme is the use, implicit or explicit, of the equidistribution properties of special points. The purpose of this paper is to give an informal exposition (see §1.1) as well as some new applications of this idea. Since our goal is merely to illustrate what can be obtained along these lines we have not tried to reach the most general results that can be obtained and, in particular, we limit ourselves to the non-vanishing problem for the family of unramified ring class characters of an imaginary quadratic field $K = \mathbb{Q}(\sqrt{-D})$ of large discriminant D .

We prove

THEOREM 1. *Let $f(z)$ be a weight 0, even, Maass (Hecke-eigen) cuspform on the modular surface $X_0(1)$; then, for any $0 < \delta < 1/2700$, one has the lower bound*

$$|\{\chi \in \widehat{\text{Cl}}_K, L(f \otimes \chi, 1/2) \neq 0\}| \gg_{\delta, f} D^\delta$$

THEOREM 2. *Let q be a prime and $f(z)$ be a holomorphic Hecke-eigen cuspform of weight 2 on $\Gamma_0(q)$ such that q remains inert in K ; then, for any $0 < \delta < 1/2700$, one has the lower bound*

$$|\{\chi \in \widehat{\text{Cl}}_K, L(f \otimes \chi, 1/2) \neq 0\}| \gg_{\delta, f} D^\delta$$

for any $\delta < 1/2700$.

The restriction to either trivial or prime level in the theorems above is merely for simplification (to avoid the occurrence of oldforms in our analysis) and extending these results to more general levels is just a technical matter. Another arguably more interesting generalization consists in considering levels q and quadratic fields K such that the sign of the functional equation is -1 : then one expects that the number of χ such that the first derivative $L'(f \otimes \chi, 1/2) \neq 0$ is $\gg D^\delta$ for some positive absolute δ . This can be proven along the above lines at least when f is holomorphic of weight 2 by using the Gross/Zagier formulas; the proof however is significantly more difficult and will be dealt with elsewhere; interestingly the proof combines the two types of equidistribution results encountered in the proof of Theorems 1 and 2 above. In the present paper, we give, for the sake of diversity, an entirely different, purely geometric, argument of such a generalization when f corresponds to an elliptic curve. For technical reasons we need to assume a certain hypothesis “ $S_{\beta, \theta}$ ” that guarantees there are enough small split primes in K . This is a fairly common feature of such problems (cf. [DFI95], [EY03]) and we regard it as almost orthogonal to the main issues we are considering. Given $\theta > 0$ and $\alpha \in]0, 1]$ we consider

HYPOTHESIS $S_{\beta, \theta}$. *The number of primitive² integral ideals \mathfrak{n} in O_K with $\text{Norm}(\mathfrak{n}) \leq D^\theta$ is $\gg D^{\beta\theta}$.*

Actually, in a sense it is remarkable that Theorems 1 and 2 above *do not* require such a hypothesis. It should be noted that $S_{\beta, \theta}$ is always true under the generalized

²That is, not divisible by any nontrivial ideal of the form (m) , with $m \in \mathbb{Z}$.

Lindelöf hypothesis and can be established unconditionally with any $\alpha \in]0, 1/3[$ for those D s whose largest prime factor is a sufficiently small power of D by the work of Graham/Ringrose [GR90] (see [DFI95] for more details).

THEOREM 3. *Assume $S_{\beta,\theta}$. Let E be an elliptic curve over \mathbb{Q} of squarefree conductor N , and suppose D is odd, coprime to N , and so that all primes dividing N split in the quadratic extension $\mathbb{Q}(\sqrt{-D})$. Then the Mordell-Weil rank of E over the Hilbert class field of $\mathbb{Q}(\sqrt{-D})$ is $\gg_{\epsilon} D^{\delta-\epsilon}$, where $\delta = \min(\beta\theta, 1/2 - 4\theta)$.*

Neither the statement nor the proof of Theorem 3 make any use of automorphic forms; but (in view of the Gross/Zagier formula) the proof actually demonstrates that the number of nonvanishing central derivatives $L'(f_E \otimes \chi, 1/2)$ is $\gg D^{\alpha}$, where f_E is the newform associated to E . Moreover, we use the ideas of the proof to give another proof (conditional on $S_{\beta,\theta}$) of Thm. 1.

We conclude the introduction by describing the main geometric issues that intervene in the proof of these Theorems. Let us consider just Theorem 1 for clarity. In that case, one has a collection of Heegner points in $\mathrm{SL}_2(\mathbb{Z}) \backslash \mathbf{H}$ with discriminant $-D$, parameterized by Cl_K . The collection of values $L(\frac{1}{2}, f \otimes \chi)$ reflects – for a fixed Maass form f , varying χ through $\widehat{\mathrm{Cl}}_K$ – the distribution of Heegner points. More precisely, it reflects the way in which the distribution of these Heegner points interacts with the subgroup structure of Cl_K . For example, if there existed a subgroup $H \subset \mathrm{Cl}_K$ such that points in the same H -coset also tend to cluster together on $\mathrm{SL}_2(\mathbb{Z}) \backslash \mathbf{H}$, this would cause the L -values to be distributed unusually. Thus, in a sense, whatever results we are able to prove about these values are (geometrically speaking) assertions that the group structure on Cl_K does not interact at all with the “proximity structure” that arises from its embedding into $\mathrm{SL}_2(\mathbb{Z}) \backslash \mathbf{H}$.

REMARK 1.1. Denote by $\mathrm{Cl}_K = \mathrm{Pic}(O_K)$ the class group of O_K and by $\widehat{\mathrm{Cl}}_K$ its dual group. We write $h_K = |\mathrm{Cl}_K| = |\widehat{\mathrm{Cl}}_K|$ for the class number of O_K . By Siegel’s theorem one has

$$(1) \quad h_K \gg_{\epsilon} D^{1/2-\epsilon}$$

(where the constant implied is not effective) so the lower bounds of Theorems 1 and 2 are far from giving a constant proportion of nonvanishing values. (In the case where f is Eisenstein, Blomer has obtained much better results: see Sec. 1.3). Moreover, both proofs make use of (1) so the constants implied are ineffective.

1.1. Nonvanishing of a single twist. Let us introduce some of the main ideas of the present paper in the most direct way, by sketching two very short proofs that *at least one* twist is nonvanishing in the context of Theorem 1. We denote by \mathbf{H} the upper-half plane. To the quadratic field $K = \mathbb{Q}(\sqrt{-D})$ – where we always assume that $-D$ is a fundamental discriminant – and each ideal class x of the maximal order O_K of $\mathbb{Q}(\sqrt{-D})$ there is associated a Heegner point $[x] \in \mathrm{SL}_2(\mathbb{Z}) \backslash \mathbf{H}$.³

One can describe the collection $He_K := \{[x] : x \in \mathrm{Cl}_K\}$ using the moduli description of $\mathrm{SL}_2(\mathbb{Z}) \backslash \mathbf{H}$: if one identifies $z \in \mathrm{SL}_2(\mathbb{Z}) \backslash \mathbf{H}$ with the isomorphism

³Namely, $[x]$ is represented by the point $\frac{-b+\sqrt{-D}}{2a}$, where $au^2 + buv + cv^2$ is a quadratic form of discriminant $-D$ corresponding to the ideal class x , i.e. there exists a fractional ideal \mathfrak{J} in the class x and a \mathbb{Z} -basis α, β for \mathfrak{J} so that $\mathrm{Norm}(u\alpha + v\beta) = \mathrm{Norm}(\mathfrak{J})(au^2 + buv + cv^2)$.

class of elliptic curves over \mathbb{C} , via $z \in \mathbf{H} \rightarrow \mathbb{C}/(\mathbb{Z} + z\mathbb{Z})$, then He_K is identified with the set of elliptic curves with CM by O_K .

If f is a Maass form and χ a character of Cl_K , one has associated a twisted L -function $L(s, f \times \chi)$, and it is known, from the work of Waldspurger and Zhang [Zha01, Zha04] that

$$(2) \quad L(f \otimes \chi, 1/2) = \frac{2}{\sqrt{D}} \left| \sum_{x \in \text{Cl}_K} \bar{\chi}(x) f([x]) \right|^2.$$

In other words: the values $L(\frac{1}{2}, f \otimes \chi)$ are the squares of the “Fourier coefficients” of the function $x \mapsto f([x])$ on the finite abelian group Cl_K . The Fourier transform being an isomorphism, in order to show that there exists *at least one* $\chi \in \widehat{\text{Cl}_K}$ such that $L(1/2, f \otimes \chi)$ is nonvanishing, it will suffice to show that $f([x]) \neq 0$ for at least one $x \in \text{Cl}_K$. There are two natural ways to approach this (for D large enough):

- (1) Probabilistically: show this is true for a random x . It is known, by a theorem of Duke, that the points $\{[x] : x \in \text{Cl}_K\}$ become equidistributed (as $D \rightarrow \infty$) w.r.t. the Riemannian measure on Y ; thus $f([x])$ is nonvanishing for a random $x \in \text{Cl}_K$.
- (2) Deterministically: show this is true for a special x . The class group Cl_K has a distinguished element, namely the identity $e \in \text{Cl}_K$; and the corresponding point $[e]$ looks very special: it lives very high in the cusp. Therefore $f([e]) \neq 0$ for obvious reasons (look at the Fourier expansion!)

Thus we have given two (fundamentally different) proofs of the fact that there exists χ such that $L(\frac{1}{2}, f \otimes \chi) \neq 0$! Soft as they appear, these simple ideas are rather powerful. The main body of the paper is devoted to quantifying these ideas further, i.e. pushing them to give that *many* twists are nonvanishing.

REMARK 1.2. The first idea is the standard one in analytic number theory: to prove that a family of quantities is nonvanishing, compute their average. It is an emerging philosophy that many averages in analytic number theory are connected to equidistribution questions and thus often to ergodic theory.

Of course we note that, in the above approach, one does not really need to know that $\{[x] : x \in \text{Cl}_K\}$ become equidistributed as $D \rightarrow \infty$; it suffices to know that this set is becoming *dense*, or even just that it is not contained in the nodal set of f . This remark is more useful in the holomorphic setting, where it means that one can use *Zariski dense* as a substitute for *dense*. See [Cor02].

In considering the second idea, it is worth keeping in mind that $f([e])$ is *extremely* small – of size $\exp(-\sqrt{D})$! We can therefore paraphrase the proof as follows: the L -function $L(\frac{1}{2}, f \otimes \chi)$ admits a certain canonical square root, which is not positive; then the sum of all these square roots is very small but known to be nonzero!

This seems of a different flavour from any analytic proof of nonvanishing known to us. Of course the central idea here – that there is always a Heegner point (in fact many) that is very high in the cusp – has been utilized in various ways before. The first example is Deuring’s result [Deu33] that the failure of the Riemann hypothesis (for ζ) would yield an effective solution to Gauss’ class number one problem; another particularly relevant application of this idea is Y. André’s lovely proof [And98] of the André–Oort conjecture for products of modular surfaces.