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The Gauss Class-Number Problems

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1. Gauss

In Articles 303 and 304 of his 1801 Disquisitiones Arithmeticae [**Gau86**], Gauss put forward several conjectures that continue to occupy us to this day. Gauss stated his conjectures in the language of binary quadratic forms (of even discriminant only, a complication that was later dispensed with). Since Dedekind's time, these conjectures have been phrased in the language of quadratic fields. This is how we will state the conjectures here, but we make some comments regarding the original versions also. Throughout this paper, $k = \mathbb{Q}(\sqrt{d})$ will be a quadratic field of discriminant d and $h(k)$ or sometimes $h(d)$ will be the class-number of k.

In Article 303, Gauss conjectures that as k runs through the complex quadratic fields (i.e., $d < 0$), $h(k) \to \infty$. He also surmises that for low class-numbers, his tables contain the complete list of fields with those class-numbers including all the one class per genus fields. This innocent addendum caused much heartache when in 1934 Heilbronn [**Hei34**] finally proved that $k(d) \to \infty$ as $d \to -\infty$ ineffectively. Thus it remained at that time impossible to even give an algorithm that would provably terminate at a predetermined time with a complete list of the complex quadratic fields of class-number one (or any other fixed class-number). By the "class-number n problem for complex quadratic fields", we mean the problem of presenting a complete list of all complex quadratic fields with class-number n. We will discuss complex quadratic fields and generalizations in Sections $3 - 5$.

For real quadratic fields (i. e., $d > 0$), Gauss surmises in Article 304 that there are infinitely many one class per genus real quadratic fields. By carrying over this surmise to prime discriminants, we get the common interpretation that Gauss conjectures there are infinitely many real quadratic fields with class-number one. We call this the "class-number one problem for real quadratic fields". This is completely unproved and, to this day, it is not even known if there are infinitely many number fields (degree arbitrary) with class-number one (or even just bounded).

We will discuss two approaches each to the one class per genus problem for complex quadratic fields and the class-number one problem for real quadratic fields. Admittedly, I don't have much hope currently for the first approaches to each problem but I think the questions raised are interesting. On the other hand, I

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think the second approaches to each problem will ultimately work. We discuss all these in Sections 4 – 6 below.

It is particularly appropriate that this paper appear in these proceedings. From Gauss and Dirichlet at the start to Landau, Siegel and Deuring, people connected with Göttingen have made major contributions to the questions discussed here.

2. Dirichlet

Dirichlet introduced L-functions in order to study the distribution of primes in progressions. A key fact in this study is that for every character χ (mod f), $L(1,\chi) \neq 0$. Dirichlet knew that

$$\prod_{\chi} L(s,\chi) = \sum_{n=1}^{\infty} a_n n^{-s} \,,$$

where the product is over all characters χ (mod f) and the a_n are non-negative integers with $a_1 = 1$. Thus for real $s > 1$ where everything converges, we must have

$$\prod_{\chi(\bmod f)} L(s,\chi) \geq 1 \,. \tag{2.1}$$

We now know that $L(s,\chi)$ has a first order pole at $s = 1$ when χ is the trivial character and is analytic at $s = 1$ for other characters. It follows from (2.1) that at most one of the $L(1,\chi)$ can be zero and that such a χ must be real since otherwise χ and $\bar{\chi}$ would both contribute zeros to the product and the product would be zero at $s = 1$. Echos of this difficulty that there could be an exceptional real χ still persist today in the study of zeros near $s = 1$.

Of necessity, Dirichlet developed his class-number formula in order to finish his theorem on primes in progressions. Although Kronecker symbols were still in the future, Dirichlet discovered that every primitive real character corresponds to a quadratic field (and conversely; the beginnings of class field theory!). We write χ_d to be the primitive real character which corresponds to $\mathbb{Q}(\sqrt{d})$. The part of the class-number formula which concerns us here gives a non-zero algebraic interpretation of $L(1,\chi_d)$. Dirichlet showed that

$$L(1,\chi_d) = \begin{cases} \dfrac{2\pi h(d)}{w_d\sqrt{|d|}} & \text{when } d < 0 \\[2ex] \dfrac{2h(d)\log(\varepsilon_d)}{\sqrt{d}} & \text{when } d > 0 \,. \end{cases}$$

Here when $d < 0$, $w_{-3} = 6$, $w_{-4} = 4$, $w_d = 2$ for $d < -4$, and when $d > 0$, ε_d is the fundamental unit of $\mathbb{Q}(\sqrt{d})$.

Landau [**Lan18b**] states that Remak made the remark that even without the class-number formula, from (2.1) we are able to see that with varying moduli there can be at most one primitive real character χ with $L(1,\chi) = 0$ and thus the primes in progressions theorem would hold outside of multiples of this one extraordinary modulus. To see this, we apply (2.1) with f the product of the conductors of the two characters χ of interest.

In truth, (2.1) also holds when the product over χ is restricted to χ running over all characters (mod f) which are identically 1 on a given subgroup of $(\mathbb{Z}/f\mathbb{Z})^*$. This is equivalent to χ running over a subgroup of the group of all characters (mod f). This product too is the zeta function of an abelian extension of $\mathbb{Q}$, but the proof that (2.1) holds does not require such knowledge. In 1918, Landau already makes use of the product in (2.1) over just four characters: the trivial character, the two interesting real characters, and their product. The product of the four L-functions is just the zeta function of the biquadratic field containing the two interesting quadratic fields.

Landau also proves that if for some constant $c > 0$, $L(s, \chi_d) \neq 0$ for real s in the range $1 - \frac{c}{\log(|d|)} < s < 1$, then

$$L(1, \chi_d) \gg \frac{1}{\log(|d|)} \quad \text{as } |d| \to \infty \ .$$

In particular, the Gauss conjectures for complex quadratic fields become consequences of the Generalized Riemann Hypothesis.

When one looks at the two 1918 Landau papers [**Lan18b**], [**Lan18a**], one is struck by how amazingly close Landau is to Siegel's 1935 theorem [**Sie35**]. All the ingredients are in the Landau papers!

3. Complex Quadratic Fields

The original Gauss class-number one conjecture is restricted to even discriminants and is much easier. For even discriminants, 2 ramifies and yet for $d > -8$, absolute value estimates show there is no integer in k with norm 2. Thus the only even class-number one discriminants are -4 and -8. Gauss also allowed non-fundamental discriminants. These correspond to ring classes and it now becomes a homework exercise to show that the non-fundamental class-number one discriminants (even or odd) are -12, -16, -27, -28.

In 1934 Heilbronn [**Hei34**] proved the Gauss Conjecture that $k(d) \to \infty$ as $d \to -\infty$. Then also in 1934, Heilbronn and Linfoot [**HL34**] proved that besides the nine known complex quadratic fields of class-number one, there is at most one more. Heilbronn's proof followed a remarkable 1933 theorem of Deuring [**Deu33**] who proved that if there were infinitely many class-number one complex quadratic fields, then the Riemann hypothesis for $\zeta(s)$ would follow! Many authors promptly carried this over to other class-numbers. But Heilbronn realized that Deuring's method would allow one to prove the generalized Riemann hypothesis for any $L(s, \chi)$ as well and this, together with Landau's earlier result above, implies Gauss's conjecture for complex quadratic fields.

These theorems are purely analytic in the sense that there is no use made of any algebraic interpretations of any special values of any relevant functions. These theorems are also noteworthy in that they are ineffective. Three decades later, the class-number one problem was solved by Baker [**Bak66**] and Stark [**Sta67**] completely. There was also the earlier discounted method of Heegner [**Hee52**] from 1952 which at the very least could be turned into a completely valid proof of the same result. It is frequently stated that my proof and Heegner's proof are the same. The two papers end up with the same Diophantine equations, but I invite anybody to read both papers and then say they give the same proof!

As an aside, I believe that I was the modern rediscoverer of Heegner's paper, having come across it in 1963 while working on my PhD thesis. Fortunately for me, if not for mathematics, it was reaffirmed at a 1963 conference in Boulder, which

I did not attend, that Heegner was incorrect and as a result I graduated in 1964 with degree in hand. Back then, it was commonly stated that the problem with Heegner's proof was that it relied on the unproved conjecture of Weber that for $d \equiv -3 \pmod 8$ and $3 \nmid d$, the classical modular function $f(z)$ evaluated at $z = \sqrt{d}$ is an algebraic integer lying in the ring class field of $k \pmod 2$. The assertion that Heegner relied upon this conjecture in his class-number one proof turned out to be absolutely false (although he did make use of Weber's conjecture in other unrelated portions of his paper) and I believe the first outline since Heegner's paper of what is actually involved in Heegner's class-number one proof occurs in my 1967 paper [**Sta67**]. In addition to Heegner [**Hee52**] and Stark [**Sta67**]. I refer the reader to Birch [**Bir69**], Deuring [**Deu68**], and Stark [**Sta69a**], [**Sta69b**]. In particular, Birch also proves Weber's conjecture. I don't think this is the place to go further into this episode.

The Gauss class-number problem for complex quadratic fields has been generalized to CM-fields (totally complex quadratic extensions of totally real fields). Since the mid 1970's we now expect that there are only finitely many CM fields with a given class-number. This has been proved effectively for normal CM fields and conditionally under each of various additional conjectures including the Generalized Riemann Hypothesis (GRH) for number field zeta functions, Artin's conjecture on L-functions being entire, and more recently under the Modified Generalized Riemann Hypothesis (MGRH) which allows real exceptions to GRH. In particular, this latter result allows Siegel zeroes to exist and would still result in effectively sending the class-number $h(K)$ of a CM field K to ∞ as K varies! It also turns out that at least some of the implied complex exceptions to GRH that hamper an attempted proof without MGRH are very near to $s = 1$. All this was prepared for a history lecture at IAS in the Fall of 1999; this part of the lecture was delivered in the Spring of 2000. It is still unpublished, but will appear someday [**Sta**].

4. Zeros of Epstein zeta functions

From the point of view of this exposition, none of the proofs of Heegner, Baker or Stark qualify as a purely analytic proof. Harder to classify is the Goldfeld [**Gol76**], Gross-Zagier [**GZ86**] combinded effective proof of the Gauss conjecture that $h(d) \to \infty$ as $d \to -\infty$. Goldfeld showed that the existence of an explicit L-function of an elliptic curve with a triple zero at $s = 1$ would imply Gauss's conjecture and Gross-Zagier prove the existence of such an L-function by giving a meaning to the first derivative at $s = 1$ of the L-function of a CM curve. For the sake of argument, I will say that this result also is not purely analytic although there remains the chance that it could be made so.

I believe that it is highly desirable that a purely analytic proof of the class-number one result be found. This is because such a proof would have a chance of extending to other fixed class-numbers and, if we were really lucky, might even begin to effectively approach the strength of Siegel's theorem. In particular, we might at long last pick up the one class per genus complex quadratic fields.

There are two potential purely analytic approaches to the class-number one problem. Both originated from the study of Epstein zeta functions. Let

$$Q(x, y) = ax^2 + bxy + cy^2 \ ,$$