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The fourth moment of Dirichlet L-functions

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Abstract. Extending a result of Heath-Brown, we prove an asymptotic formula for the fourth moment of $L(\frac{1}{2},\chi)$ where χ ranges over the primitive Dirichlet characters (mod q).

1. Introduction

In [**HB81**], D.R. Heath-Brown showed that

$$\sum_{\chi \pmod q}^{*} |L(\tfrac{1}{2},\chi)|^4 = \frac{\varphi^*(q)}{2\pi^2}\prod_{p|q}\frac{(1-p^{-1})^3}{(1+p^{-1})}(\log q)^4 + O(2^{\omega(q)}q(\log q)^3). \tag{1.1}$$

Here $\sum^*$ denotes summation over primitive characters χ (mod q), $\varphi^*(q)$ denotes the number of primitive characters (mod q), and $\omega(q)$ denotes the number of distinct prime factors of q. Note that $\varphi^*(q)$ is a multiplicative function given by $\varphi^*(p) = p-2$ for primes p, and $\varphi^*(p^k) = p^k(1-1/p)^2$ for $k \geq 2$ (see Lemma 1 below). Also note that when $q \equiv 2 \pmod 4$ there are no primitive characters (mod q), and so below we will assume that $q \not\equiv 2 \pmod 4$. For $q \not\equiv 2 \pmod 4$ it is useful to keep in mind that the main term in (1.1) is $\asymp q(\varphi(q)/q)^6(\log q)^4$.

Heath-Brown's result represents a q-analog of Ingham's fourth moment for $\zeta(s)$:

$$\int_0^T |\zeta(\tfrac{1}{2}+it)|^4 dt \sim \frac{T}{2\pi^2}(\log T)^4.$$

When $\omega(q) \leq (1/\log 2 - \epsilon)\log\log q$ (which holds for almost all q) the error term in (1.1) is dominated by the main term and (1.1) gives the q-analog of Ingham's result. However if q is even a little more than 'ordinarily composite', with $\omega(q) \geq (\log\log q)/\log 2$, then the error term in (1.1) dominates the main term. In this note we remedy this, and obtain an asymptotic formula valid for all large q.

Theorem. *For all large q we have*

$$\sum_{\chi \pmod q}^{*} |L(\tfrac{1}{2},\chi)|^4 = \frac{\varphi^*(q)}{2\pi^2}\prod_{p|q}\frac{(1-p^{-1})^3}{(1+p^{-1})}(\log q)^4\Big(1+O\Big(\frac{\omega(q)}{\log q}\sqrt{\frac{q}{\varphi(q)}}\Big)\Big)+O(q(\log q)^{\frac{7}{2}}).$$

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Since $\omega(q) \ll \log q / \log\log q$, and $q/\varphi(q) \ll \log\log q$, we see that

$$(\omega(q)/\log q)\sqrt{q/\varphi(q)} \ll 1/\sqrt{\log\log q}.$$

Thus our Theorem gives a genuine asymptotic formula for all large q.

For any character $\chi \pmod q$ (not necessarily primitive) let $\mathfrak{a} = 0$ or 1 be given by $\chi(-1) = (-1)^{\mathfrak{a}}$. For $x > 0$ we define

$$W_{\mathfrak{a}}(x) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \Big(\frac{\Gamma(\frac{s+\frac12+\mathfrak{a}}{2})}{\Gamma(\frac{\frac12+\mathfrak{a}}{2})}\Big)^2 x^{-s}\frac{ds}{s}, \tag{1.2}$$

for any positive c. By moving the line of integration to $c = -\frac12 + \epsilon$ we may see that

$$W(x) = 1 + O(x^{\frac12-\epsilon}), \tag{1.3a}$$

and from the definition (1.2) we also get that

$$W(x) = O_c(x^{-c}). \tag{1.3b}$$

We define

$$A(\chi) := \sum_{a,b=1}^{\infty} \frac{\chi(a)\overline{\chi}(b)}{\sqrt{ab}} W_{\mathfrak{a}}\Big(\frac{\pi ab}{q}\Big). \tag{1.4}$$

If χ is primitive then $|L(\frac12,\chi)|^2 = 2A(\chi)$ (see Lemma 2 below). Let $Z = q/2^{\omega(q)}$ and decompose $A(\chi)$ as $B(\chi) + C(\chi)$ where

$$B(\chi) = \sum_{\substack{a,\, b \ge 1 \\ ab \le Z}} \frac{\chi(a)\overline{\chi}(b)}{\sqrt{ab}} W_{\mathfrak{a}}\Big(\frac{\pi ab}{q}\Big),$$

and

$$C(\chi) = \sum_{\substack{a,\, b \ge 1 \\ ab > Z}} \frac{\chi(a)\overline{\chi}(b)}{\sqrt{ab}} W_{\mathfrak{a}}\Big(\frac{\pi ab}{q}\Big).$$

Our main theorem will follow from the following two Propositions.

Proposition 1. *We have*

$$\sideset{}{^*}\sum_{\chi \pmod q} |B(\chi)|^2 = \frac{\varphi^*(q)}{8\pi^2}\prod_{p|q}\frac{(1-1/p)^3}{(1+1/p)}(\log q)^4\Big(1 + O\Big(\frac{\omega(q)}{\log q}\Big)\Big).$$

Proposition 2. *We have*

$$\sum_{\chi \pmod q} |C(\chi)|^2 \ll q\Big(\frac{\varphi(q)}{q}\Big)^5(\omega(q)\log q)^2 + q(\log q)^3.$$

Proof of the Theorem. Since $|L(\frac12,\chi)|^2 = 2A(\chi) = 2(B(\chi)+C(\chi))$ for primitive characters χ we have

$$\sideset{}{^*}\sum_{\chi \pmod q} |L(\tfrac12,\chi)|^4 = 4\sideset{}{^*}\sum_{\chi \pmod q}\Big(|B(\chi)|^2 + 2B(\chi)C(\chi) + |C(\chi)|^2\Big).$$

The first and third terms on the right hand side are handled directly by Propositions 1 and 2. By Cauchy's inequality

$$\sideset{}{^*}\sum_{\chi \pmod q} |B(\chi)C(\chi)| \le \Big(\sideset{}{^*}\sum_{\chi \pmod q}|B(\chi)|^2\Big)^{\frac12}\Big(\sum_{\chi \pmod q}|C(\chi)|^2\Big)^{\frac12},$$

and thus Propositions 1 and 2 furnish an estimate for the second term also. Combining these results gives the Theorem. □

In [**HB79**], Heath-Brown refined Ingham's fourth moment for $\zeta(s)$, and obtained an asymptotic formula with a remainder term $O(T^{\frac{7}{8}+\epsilon})$. It remains a challenging open problem to obtain an asymptotic formula for $\sum^*_{\chi \pmod q} |L(\frac12,\chi)|^4$ where the error term is $O(q^{1-\delta})$ for some positive δ.

This note arose from a conversation with Roger Heath-Brown at the Gauss-Dirichlet conference where he reminded me of this problem. It is a pleasure to thank him for this and other stimulating discussions.

2. Lemmas

Lemma 1. *If* $(r,q)=1$ *then*

$$\sideset{}{^*}\sum_{\chi \pmod q} \chi(r) = \sum_{k|(q,r-1)} \varphi(k)\mu(q/k).$$

Proof. If we write $h_r(k) = \sum^*_{\chi \pmod k} \chi(r)$ then for $(r,q)=1$ we have

$$\sum_{k|q} h_r(k) = \sum_{\chi \pmod q} \chi(r) = \begin{cases} \varphi(q) & \text{if } q \mid r-1 \\ 0 & \text{otherwise.}\end{cases}$$

The Lemma now follows by Möbius inversion. □

Note that taking $r=1$ gives the formula for $\varphi^*(q)$ given in the introduction. If we restrict attention to characters of a given sign $\mathfrak{a}$ then we have, for $(mn,q)=1$,

(2.1)
$$\sideset{}{^*}\sum_{\substack{\chi \pmod q \\ \chi(-1)=(-1)^{\mathfrak{a}}}} \chi(m)\overline{\chi}(n) = \frac12 \sum_{k|(q,|m-n|)} \varphi(k)\mu(q/k) + \frac{(-1)^{\mathfrak{a}}}{2} \sum_{k|(q,m+n)} \varphi(k)\mu(q/k).$$

Lemma 2. *If* χ *is a primitive character* $\pmod q$ *with* $\chi(-1)=(-1)^{\mathfrak{a}}$ *then*

$$|L(\tfrac12,\chi)|^2 = 2A(\chi),$$

where $A(\chi)$ *is defined in (1.4).*

Proof. We recall the functional equation (see Chapter 9 of [**Dav00**])

$$\Lambda(\tfrac12+s,\chi) = \Big(\frac{q}{\pi}\Big)^{s/2}\Gamma\Big(\frac{s+\frac12+\mathfrak{a}}{2}\Big)L(\tfrac12+s,\chi) = \frac{\tau(\chi)}{i^{\mathfrak{a}}\sqrt{q}}\Lambda(\tfrac12-s,\overline{\chi}),$$

which yields

$$\Lambda(\tfrac12+s,\chi)\Lambda(\tfrac12+s,\overline{\chi}) = \Lambda(\tfrac12-s,\chi)\Lambda(\tfrac12-s,\overline{\chi}). \tag{2.2}$$

For $c>\frac12$ we consider

$$I := \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{\Lambda(\frac12+s,\chi)\Lambda(\frac12+s,\overline{\chi})}{\Gamma(\frac{\frac12+\mathfrak{a}}{2})^2}\frac{ds}{s}.$$

We move the line of integration to $\mathrm{Re}(s)=-c$, and use the functional equation (2.2). This readily gives that $I = |L(\frac12,\chi)|^2 - I$, so that $|L(\frac12,\chi)|^2 = 2I$. On the other hand, expanding $L(\frac12+s,\chi)L(\frac12+s,\overline{\chi})$ into its Dirichlet series and integrating termwise, we get that $I = A(\chi)$. This proves the Lemma. □

We shall require the following bounds for divisor sums. If k and ℓ are positive integers with $\ell k \ll x^{\frac{5}{4}}$ then

$$\sum_{\substack{n \le x \\ (n,k)=1}} d(n)d(\ell k \pm n) \ll x(\log x)^2 \sum_{d|\ell} d^{-1}, \tag{2.3}$$

provided that $x \le \ell k$ if the negative sign holds. This is given in (17) of Heath-Brown [**HB81**]. Secondly, we record a result of P. Shiu [**Shi80**] which gives that

$$\sum_{\substack{n \le x \\ n \equiv r \pmod k}} d(n) \ll \frac{\varphi(k)}{k^2} x \log x, \tag{2.4}$$

where $(r,k)=1$ and $x \ge k^{1+\delta}$ for some fixed $\delta > 0$.

Lemma 3. *Let k be a positive integer, and let Z_1 and Z_2 be real numbers ≥ 2. If $Z_1Z_2 > k^{\frac{19}{10}}$ then*

$$\sum_{\substack{Z_1 \le ab < 2Z_1 \\ Z_2 \le cd < 2Z_2 \\ (abcd,k)=1 \\ ac \equiv \pm bd \pmod k \\ ac \ne bd}} 1 \ll \frac{Z_1Z_2}{k}(\log(Z_1Z_2))^3.$$

If $Z_1Z_2 \le k^{\frac{19}{10}}$ the quantity estimated above is $\ll (Z_1Z_2)^{1+\epsilon}/k$.

Proof. By symmetry we may just focus on the terms with $ac > bd$. Write $n = bd$ and $ac = k\ell \pm bd$. Note that $k\ell \le 2ac$ and so $1 \le \ell \le 8Z_1Z_2/k$. Moreover since $ac \ge k\ell/2$ we have that $bd \le 4Z_1Z_2/(ac) \le 8Z_1Z_2/(k\ell)$. Thus the sum we desire to estimate is

$$\ll \sum_{1 \le \ell \le 8Z_1Z_2/k} \sum_{\substack{n \le 8Z_1Z_2/(k\ell) \\ n < k\ell \pm n \\ (n,k)=1}} d(n)d(k\ell \pm n). \tag{2.5}$$

Since $d(n)d(k\ell \pm n) \ll (Z_1Z_2)^\epsilon$ the second assertion of the Lemma follows.

Now suppose that $Z_1Z_2 > k^{\frac{19}{10}}$. We distinguish the cases $k\ell \le (Z_1Z_2)^{\frac{11}{20}}$ and $k\ell > (Z_1Z_2)^{\frac{11}{20}}$. In the first case we estimate the sum over n using (2.3). Thus such terms contribute to (2.5)

$$\ll \sum_{\ell \le (Z_1Z_2)^{\frac{11}{20}}/k} \frac{Z_1Z_2}{k\ell}(\log Z_1Z_2)^2 \sum_{d|\ell} d^{-1} \ll \frac{Z_1Z_2}{k}(\log Z_1Z_2)^3.$$

Now consider the second case. Here we sum over ℓ first. Writing $m = k\ell \pm n (= ac)$ we see that such terms contribute

$$\ll \sum_{n \le 8Z_1Z_2/k} d(n) \sum_{\substack{(Z_1Z_2)^{\frac{11}{20}}/2 \le m \le 4Z_1Z_2/n \\ m \equiv \pm n \pmod k}} d(m),$$

and by (2.4) (which applies as $(Z_1Z_2)^{\frac{11}{20}} > k^{\frac{209}{200}}$) this is

$$\ll \sum_{n \le 8Z_1Z_2/k} d(n) \frac{Z_1Z_2}{kn} \log Z_1Z_2 \ll \frac{Z_1Z_2}{k}(\log Z_1Z_2)^3.$$

The proof is complete. $\square$