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Reciprocal Geodesics

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ABSTRACT. The closed geodesics on the modular surface which are equivalent to themselves when their orientation is reversed have recently arisen in a number of different contexts.We examine their relation to Gauss' ambiguous binary quadratic forms and to elements of order four in his composition groups.We give a parametrization of these geodesics and use this to count them asymptotically and to investigate their distribution.

This note is concerned with parametrizing, counting and equidistribution of conjugacy classes of infinite maximal dihedral subgroups of $\Gamma = PSL(2, \mathbb{Z})$ and their connection to Gauss' ambiguous quadratic forms. These subgroups feature in the recent work of Connolly and Davis on invariants for the connect sum problem for manifolds [**CD**]. They also come up in [**PR04**] (also see the references therein) in connection with the stability of kicked dynamics of torus automorphisms as well as in the theory of quasimorphisms of Γ. In [**GS80**] they arise when classifying codimension one foliations of torus bundles over the circle. Apparently they are of quite wide interest. As pointed out to me by Peter Doyle, these conjugacy classes and the corresponding reciprocal geodesics, are already discussed in a couple of places in the volumes of Fricke and Klein ([**FK**], Vol. I, page 269, Vol II, page 165). The discussion below essentially reproduces a (long) letter that I wrote to Jim Davis (June, 2005).

Denote by $\{\gamma\}_\Gamma$ the conjugacy class in Γ of an element $\gamma \in \Gamma$. The elliptic and parabolic classes (i.e., those with $t(\gamma) \leq 2$ where $t(\gamma) = |\text{trace } \gamma|$) are well-known through examining the standard fundamental domain for Γ as it acts on $\mathbb{H}$. We restrict our attention to hyperbolic γ's and we call such a γ primitive (or prime) if it is not a proper power of another element of Γ. Denote by P the set of such elements and by Π the corresponding set of conjugacy classes. The primitive elements generate the maximal hyperbolic cyclic subgroups of Γ. We call a $p \in P$ reciprocal if $p^{-1} = S^{-1}pS$ for some $S \in \Gamma$. In this case, $S^2 = 1$ (proofs of this and further claims are given below) and S is unique up to multiplication on the left by $\gamma \in \langle p \rangle$. Let R denote the set of such reciprocal elements. For $r \in R$ the group $D_r = \langle r, S \rangle$, depends only on r and it is a maximal infinite dihedral subgroup of

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Γ. Moreover, all of the latter arise in this way. Thus, the determination of the conjugacy classes of these dihedral subgroups is the same as determining ρ, the subset of Π consisting of conjugacy classes of reciprocal elements. Geometrically, each $p \in P$ gives rise to an oriented primitive closed geodesic on $\Gamma\backslash\mathbb{H}$, whose length is $\log N(p)$ where $N(p) = \left[\left(t(p) + \sqrt{t(p)^2 - 4}\right)/2\right]^2$. Conjugate elements give rise to the same oriented closed geodesic. A closed geodesic is equivalent to itself with its orientation reversed iff it corresponds to an $\{r\} \in \rho$.

The question as to whether a given γ is conjugate to γ^{-1} in Γ is reflected in part in the corresponding local question. If $p \equiv 3 \pmod 4$, then $c = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$ is not conjugate to c^{-1} in $SL(2, \mathbb{F}_p)$, on the other hand, if $p \equiv 1 \pmod 4$ then every $c \in SL(2, \mathbb{F}_p)$ is conjugate to c^{-1}. This difficulty of being conjugate in $G(\bar{F})$ but not in $G(F)$ does not arise if $G = GL_n$ (F a field) and it is the source of a basic general difficulty associated with conjugacy classes in G and the (adelic) trace formula and its stabilization [**Lan79**]. For the case at hand when working over $\mathbb{Z}$, there is the added issue associated with the lack of a local to global principle and in particular the class group enters. In fact, certain elements of order dividing four in Gauss' composition group play a critical role in the analysis of the reciprocal classes.

In order to study ρ it is convenient to introduce some other set theoretic involutions of Π. Let ϕ_R be the involution of Γ given by $\phi_R(\gamma) = \gamma^{-1}$. Let $\phi_w(\gamma) = w^{-1}\gamma w$ where $w = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \in PGL(2, \mathbb{Z})$ (modulo inner automorphism ϕ_w generates the outer automorphisms of Γ coming from $PGL(2, \mathbb{Z})$). ϕ_R and ϕ_w commute and set $\phi_A = \phi_R \circ \phi_w = \phi_w \circ \phi_R$. These three involutions generate the Klein group G of order 4. The action of G on Γ preserves P and Π. For H a subgroup of G, let $\Pi_H = \{\{p\} \in \Pi : \phi(\{p\}) = \{p\} \text{ for } \phi \in H\}$. Thus $\Pi_{\{e\}} = \Pi$ and $\Pi_{\langle \phi_R \rangle} = \rho$. We call the elements in $\Pi_{\langle \phi_A \rangle}$ ambiguous classes (we will see that they are related to Gauss' ambiguous classes of quadratic forms) and of $\Pi_{\langle \phi_w \rangle}$, inert classes. Note that the involution $\gamma \to \gamma^t$ is, up to conjugacy in Γ, the same as ϕ_R, since the contragredient satisfies ${}^t g^{-1} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} g \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$. Thus $p \in P$ is reciprocal iff p is conjugate to p^t.

To give an explicit parametrization of ρ let

$$C = \left\{(a, b) \in \mathbb{Z}^2 : (a, b) = 1,\, a > 0,\, d = 4a^2 + b^2 \text{ is not a square}\right\}. \tag{1}$$

To each $(a, b) \in C$ let (t_0, u_0) be the least solution with $t_0 > 0$ and $u_0 > 0$ of the Pell equation

$$t^2 - du^2 = 4. \tag{2}$$

Define $\psi : C \longrightarrow \rho$ by

$$(a, b) \longrightarrow \left\{ \begin{bmatrix} \dfrac{t_0 - bu_0}{2} & au_0 \\ au_0 & \dfrac{t_0 + bu_0}{2} \end{bmatrix} \right\}_\Gamma , \tag{3}$$

It is clear that $\psi((a,b))$ is reciprocal since an $A \in \Gamma$ is symmetric iff $S_0^{-1} A S_0 = A^{-1}$ where $S_0 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$. Our central assertion concerning parametrizing ρ is;

PROPOSITION 1. *$\psi : C \longrightarrow \rho$ is two-to-one and onto.* *

There is a further stratification to the correspondence (3). Let

(4) $$\mathcal{D} = \{ m \,|\, m > 0 \,,\, m \equiv 0,1 \pmod 4 \,, m \text{ not a square} \} \,.$$

Then

$$C = \bigcup_{d \in \mathcal{D}} C_d$$

where

(5) $$C_d = \left\{ (a,b) \in C \,|\, 4a^2 + b^2 = d \right\} \,.$$

Elementary considerations concerning proper representations of integers as a sum of two squares shows that C_d is empty unless d has only prime divisors p with $p \equiv 1 \pmod 4$ or the prime 2 which can occur to exponent $\alpha = 0, 2$ or 3. Denote this subset of $\mathcal{D}$ by $\mathcal{D}_R$. Moreover for $d \in \mathcal{D}_R$,

(6) $$|C_d| = 2\,\nu(d)$$

where for any $d \in \mathcal{D}, \nu(d)$ is the number of genera of binary quadratic forms of discriminant d ((6) is not a coincidence as will be explained below). Explicitly $\nu(d)$ is given as follows: If $d = 2^\alpha D$ with D odd and if λ is the number of distinct prime divisors of D then

(6′) $$\nu(d) = \begin{cases} 2^{\lambda-1} & \text{if} \quad \alpha = 0 \\ 2^{\lambda-1} & \text{if} \quad \alpha = 2 \quad \text{and} \quad D \equiv 1 \pmod 4 \\ 2^{\lambda} & \text{if} \quad \alpha = 2 \quad \text{and} \quad D \equiv 3 \pmod 4 \\ 2^{\lambda} & \text{if} \quad \alpha = 3 \quad \text{or} \quad 4 \\ 2^{\lambda+1} & \text{if} \quad \alpha \geq 5 \,. \end{cases}$$

Corresponding to (5) we have

(7) $$\rho = \bigsqcup_{d \in \mathcal{D}_R} \rho_d \,,$$

with $\rho_d = \psi(C_d)$. In particular, $\psi : C_d \longrightarrow \rho_d$ is two-to-one and onto and hence

(8) $$|\rho_d| = \nu(d) \text{ for } d \in \mathcal{D}_R \,.$$

Local considerations show that for $d \in \mathcal{D}$ the Pell equation

(9) $$t^2 - du^2 = -4 \,,$$

can only have a solution if $d \in \mathcal{D}_R$. When $d \in \mathcal{D}_R$ it may or may not have a solution. Let $\mathcal{D}_R^-$ be those d's for which (9) has a solution and $\mathcal{D}_R^+$ the set of $d \in \mathcal{D}_R$ for which (9) has no integer solution. Then

(i) For $d \in \mathcal{D}_R^+$ none of the $\{r\} \in \rho_d$, are ambiguous.
(ii) For $d \in \mathcal{D}_R^-$, every $\{r\} \in \rho_d$ is ambiguous.

*Part of this Proposition is noted in ([**FK**], Vol. I, pages 267-269).

In this last case (ii) we can choose an explicit section of the two-to-one map (3). For $d \in \mathcal{D}_R^-$ let $C_d^- = \{(a,b) : b < 0\}$, then $\psi : C_d^- \longrightarrow \rho_d$ is a bijection.[†]

Using these parameterizations as well as some standard techniques from the spectral theory of $\Gamma\backslash\mathbb{H}$ one can count the number of primitive reciprocal classes. We order the primes $\{p\} \in \Pi$ by their trace $t(p)$ (this is equivalent to ordering the corresponding prime geodesics by their lengths). For H a subgroup of G and $x > 2$ let

$$\Pi_H(x) := \sum_{\substack{\{p\} \in \Pi_H \\ t(p) \leq x}} 1 \,. \tag{10}$$

THEOREM 2. *As $x \longrightarrow \infty$ we have the following asymptotics:*

$$\Pi_{\{1\}}(x) \sim \frac{x^2}{2\log x}\,, \tag{11}$$

$$\Pi_{\langle \phi_A \rangle}(x) \sim \frac{97}{8\pi^2}\, x(\log x)^2\,, \tag{12}$$

$$\Pi_{\langle \phi_R \rangle}(x) \sim \frac{3}{8}\, x\,, \tag{13}$$

$$\Pi_{\langle \phi_w \rangle}(x) \sim \frac{x}{2\log x} \tag{14}$$

and

$$\Pi_G(x) \sim \frac{21}{8\pi}\, x^{1/2} \log x\,. \tag{15}$$

(All of these are established with an exponent saving for the remainder).

In particular, roughly the square root of all the primitive classes are reciprocal while the fourth root of them are simultaneously reciprocal ambiguous and inert.

We turn to the proofs of the above statements as well as a further discussion connecting ρ with elements of order dividing four in Gauss' composition groups.

We begin with the implication $S^{-1}pS = p^{-1} \Longrightarrow S^2 = 1$. This is true already in $PSL(2,\mathbb{R})$. Indeed, in this group p is conjugate to $\pm \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}$ with $\lambda > 1$. Hence $Sp^{-1} = pS$ with $S = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \Longrightarrow a = d = 0$, i.e., $S = \pm \begin{bmatrix} 0 & \beta \\ -\beta^{-1} & 0 \end{bmatrix}$ and so $S^2 = 1$. If S and S_1 satisfy $x^{-1}px = p^{-1}$ then $SS_1^{-1} \in \Gamma_p$ the centralizer of p in Γ. But $\Gamma_p = \langle p \rangle$ and hence $S = \beta S_1$ with $\beta \in \langle p \rangle$. Now every element $S \in \Gamma$ whose order is two (i.e., an elliptic element of order 2) is conjugate in Γ to $S_0 = \pm \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$. Hence any $r \in R$ is conjugate to an element $\gamma \in \Gamma$ for which $S_0^{-1}\gamma S_0 = \gamma^{-1}$. The last is equivalent to γ being symmetric. Thus each $r \in R$ is conjugate to a $\gamma \in R$ with $\gamma = \gamma^t$. (15′)

We can be more precise:

LEMMA 3. *Every $r \in R$ is conjugate to exactly four γ's which are symmetric.*

[†]For a general $d \in \mathcal{D}_R^+$ it appears to be difficult to determine explicitly a one-to-one section of ψ.