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Rational points of bounded height on threefolds

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Abstract. Let $n_{e,f}(B)$ be the number of non-trivial positive integer solutions $x_0, x_1, x_2, y_0, y_1, y_2 \leq B$ to the simultaneous equations

$$x_0^e + x_1^e + x_2^e = y_0^e + y_1^e + y_2^e, \qquad x_0^f + x_1^f + x_2^f = y_0^f + y_1^f + y_2^f.$$

We show that $n_{1,4}(B) = O_\varepsilon(B^{85/32+\varepsilon}), n_{1,5}(B) = O_\varepsilon(B^{51/20+\varepsilon})$ and that $n_{e,f}(B) = O_{e,f,\varepsilon}(B^{5/2+\varepsilon})$ if $ef \geq 6$ and $f \geq 4$. These estimates are deduced from general upper bounds for the number of rational points of bounded height on projective threefolds over $\mathbb{Q}$.

Introduction

This paper deals with the number $N(X, B)$ of rational points of height at most B on projective threefolds $X \subset \mathbb{P}^n$ over $\mathbb{Q}$. To define the height $H(x)$ of a rational point x on $\mathbb{P}^n$, we choose a primitive integral $(n+1)$-tuple $(x_0, ..., x_n)$ representing x and let $H(x) = \max(|x_0|, ..., |x_n|)$. Our main result is the following.

Theorem 0.1. *Let $X \subset \mathbb{P}^n$ be a geometrically integral projective threefold over $\mathbb{Q}$ of degree d and let X' be the complement of the union of all planes on X. Then*

$$N(X', B) = \begin{cases} O_{n,\varepsilon}(B^{15\sqrt{3}/16+5/4+\varepsilon}) & \text{if} \quad d = 3, \\ O_{n,\varepsilon}(B^{1205/448+\varepsilon}) & \text{if} \quad d = 4, \\ O_{n,\varepsilon}(B^{51/20+\varepsilon}) & \text{if} \quad d = 5, \\ O_{d,n,\varepsilon}(B^{5/2+\varepsilon}) & \text{if} \quad d \geq 6. \end{cases}$$

If $n = d = 4$ and X is not a cone of a Steiner surface, then

$$N(X', B) = O_{n,\varepsilon}(B^{85/32+\varepsilon}).$$

This bound is better than the bound $O_{d,n,\varepsilon}(B^{11/4+\varepsilon} + B^{5/2+5/3d+\varepsilon})$ in [**Salc**], §8. An important special case is the following.

Theorem 0.2. *Let $(a_0, \ldots, a_5)$ and $(b_0, \ldots, b_5)$ be two sextuples of rational numbers all different from zero and $e < f$ be positive integers. Let $X \subset \mathbb{P}^5$ be the threefold defined by the two equations $a_0x_0^e + \ldots + a_5x_5^e = 0$ and $b_0x_0^f + \ldots + b_5x_5^f = 0$. Then there are only finitely many planes on X if $f \geq 3$. Moreover, if $X' \subset X$ is*

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the complement of these planes in X, then

$$N(X',B)=\begin{cases} O_\varepsilon(B^{15\sqrt{3}/16+5/4+\varepsilon}) & \text{if} \quad e=1 \text{ and } f=3,\\ O_\varepsilon(B^{85/32+\varepsilon}) & \text{if} \quad e=1 \text{ and } f=4,\\ O_\varepsilon(B^{51/20+\varepsilon}) & \text{if} \quad e=1 \text{ and } f=5,\\ O_{e,f,\varepsilon}(B^{5/2+\varepsilon}) & \text{if} \quad ef\geq 6. \end{cases}$$

As a corollary we obtain from Lemma 1 in [**BHB**] the following result on pairs of simultaneous equal sums of three powers.

COROLLARY 0.3. Let $n_{e,f}(B)$, $e<f$ be the number of solutions in positive integers x_i, $y_i \leq B$ to the two polynomial equations

$$\begin{aligned} x_0^e+x_1^e+x_2^e &= y_0^e+y_1^e+y_2^e\\ x_0^f+x_1^f+x_2^f &= y_0^f+y_1^f+y_2^f \end{aligned}$$

where $(x_0,x_1,x_2)\neq(y_i,y_j,y_k)$ for all six permutations of (i,j,k) of $(0,1,2)$. Then,

$$\begin{aligned} &n_{1,4}(B)=O_\varepsilon(B^{85/32+\varepsilon})\\ &n_{1,5}(B)=O_\varepsilon(B^{51/20+\varepsilon})\\ &n_{e,f}(B)=O_{e,f,\varepsilon}(B^{5/2+\varepsilon}) \quad \text{if} \;\; ef\geq 6 \text{ and } f\geq 4. \end{aligned}$$

Previously, it has been shown by Greaves [**Gre97**] that $n_{1,f}(B)=O_\varepsilon(B^{17/6+\varepsilon})$ and by Skinner-Wooley[**SW97**] that $n_{1,f}(B)=O_\varepsilon(B^{8/3+1/(f-1)+\varepsilon})$. Moreover, work of Wooley [**Woo96**] shows that $n_{2,3}(B)=O_\varepsilon(B^{7/3+\varepsilon})$ and Tsui and Wooley [**TW99**] have shown that $n_{2,4}(B)=O_\varepsilon(B^{36/13+\varepsilon})$. Finally, one may find the estimate

$$n_{e,f}(B)=O_{e,f,\varepsilon}(B^{11/4+\varepsilon}+B^{5/2+5/3ef+\varepsilon})$$

in the paper of Browning and Heath-Brown [**BHB**]. Our estimate for $n_{e,f}(B)$ is superior to the previous estimates when $f\geq 4$.

The main idea of the proof of Theorem 0.1 is to use hyperplane sections to reduce to counting problems for surfaces. For the geometrically integral hyperplane sections we use thereby the new sharp estimates for surfaces in [**Sala**].

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1. The hyperplane sections given by Siegel's lemma

Let $G_r(\mathbb{P}^n)$ be the Grassmannian of r-planes on $\mathbb{P}^n$. It is embedded into $\mathbb{P}^{\binom{n+1}{r+1}-1}$ by the Plücker embedding. In particular, if $r=n-1$, then we may identify $G_r(\mathbb{P}^n)$ with the dual projective space $\mathbb{P}^{n\vee}$. The height $H(\Lambda)$ of a rational r-plane $\Lambda\subset\mathbb{P}^n$ is by definition the height of its Plücker coordinates. In particular, if $r=n-1$ then the height of a hyperplane $\Lambda\subset\mathbb{P}^n$ defined by $c_0x_0+\ldots+c_nx_n=0$, is the height of the rational point $(c_0,\ldots,c_n)$ in $\mathbb{P}^{n\vee}$.

In order to prove Theorem 0.1 for hypersurfaces in $\mathbb{P}^4$, we shall need the following two lemmas from the geometry of numbers. See [**Sch91**], Chap I, for example.

LEMMA 1.1. *Let x be a rational point of height $\leq B$ on $\mathbb{P}^4$. Then x lies on a hyperplane Π of height $H(\Pi)\leq(5B)^{1/4}$.*

LEMMA 1.2. *There is an absolute constant κ such that $H(\Pi)\leq\kappa H(\Lambda)^{1/3}$ for any rational hyperplane Π of minimal height containing a given line $\Lambda\subset\mathbb{P}^4$.*

We now introduce the following notation for a geometrically integral hypersurface $X \subset \mathbb{P}^4$.

NOTATION 1.3. (i) *X' is the complement of the union of all planes on X.*
(ii) *$S(X,B)$ is the set of all rational points of height at most B on X.*
(iii) *$P(X,B)$ is the set of all rational planes $\Theta \subset X$ which are spanned by their rational points of height $\leq B$ and which are contained in a rational hyperplane $\Pi \subset \mathbb{P}^4$ of height $H(\Pi) \leq (5B)^{1/4}$.*
(iv) *$\widetilde{S}(X,B)$ is the set of all rational points of height at most B on X, which do not lie on a plane $\Theta \subset X$ in $P(X,B)$.*
(v) *$N(X,B) = \#S(X,B)$ and $\widetilde{N}(X,B) = \#\widetilde{S}(X,B)$.*

2. The hyperplane sections which are not geometrically integral

We shall in this section estimate the contribution to $\widetilde{N}(X,B)$ from the hyperplane sections $\Pi \cap X$, which are not geometrically integral.

LEMMA 2.1. *Let $X \subset \mathbb{P}^4$ be a geometrically integral projective threefold of degree $d \geq 2$ over some field. Let $\mathbb{P}^{4\vee}$ be the dual projective space parameterising hyperplanes $\Pi \subset \mathbb{P}^4$ and let $c < d$ be a positive integer. Then the following holds.*

a) *There is a closed subscheme $W_{c,d} \subset \mathbb{P}^{4\vee}$ which parameterises the hyperplanes Π such that $\Pi \cap X$ contains a surface of degree c. The sum of the degrees of the irreducible components of $W_{c,d}$ is bounded in terms of d.*
b) $\dim W_{c,d} \leq 2$.
c) *If there is a plane on $W_{c,d} \subset \mathbb{P}^{4\vee}$, then X is a cone over a curve.*

PROOF. a) See [**Sal05**], Lemma 3.3.

(b) There exists by the theorem of Bertini a hyperplane $\Pi_0 \subset \mathbb{P}^{4\vee}$ and a plane $\Theta \subset \Pi_0$ such that $\Pi_0 \cap X$ and $\Theta \cap X$ are geometrically integral. Let $\Pi_0^\vee$ be the dual projective 3-space of Π_0 which parameterises all planes in Π_0 and $f : W_{c,d} \to \Pi_0^\vee$ be the linear morphism which sends the Grassmann point of $\Pi \subset \mathbb{P}^4$ to the Grassmann points of $\Pi \cap \Pi_0 \subset \Pi_0$. Then f must be finite since otherwise one of the fibres of f would contain a line passing through the Grassmann point of $\Pi_0 \subset \mathbb{P}^4$. Also, f is not surjective since Θ cannot be of the form $\Pi \cap \Pi_0$ for any hyperplane $\Pi \subset \mathbb{P}^4$ parameterised by a point in $W_{c,d}$. Hence $\dim W_{c,d} = \dim f(W_{c,d}) \leq \dim \Pi_0 - 1 = 2$.

(c) Let $\Gamma \subset \mathbb{P}^{4\vee}$ be a plane, $\Lambda \subset \mathbb{P}^4$ the dual line, $\pi : Z \to \mathbb{P}^4$ the blow-up at Λ and $\widetilde{X} = \pi^{-1}(X)$. Let $p : \mathbb{P}^4 \backslash \Lambda \to \mathbb{P}^2$ be a linear projection from Λ and $q : Z \to \mathbb{P}^2$ the morphism induced by p. If $q(\widetilde{X}) \neq \mathbb{P}^2$, then X is a cone over a curve with Λ as vertex. If $q(\widetilde{X}) = \mathbb{P}^2$, then we apply the theorem of Bertini to the restriction of q to $\widetilde{X}$. This implies that $q^{-1}(L) \cap \widetilde{X}$ is geometrically integral for a generic line $L \subset \mathbb{P}^2$. Let $\Pi \subset \mathbb{P}^4$ be the hyperplane given by the closure of $p^{-1}(L)$. Then $\Pi \cap X$ is geometrically integral since $q^{-1}(L) \cap \widetilde{X}$ is mapped birationally onto $\Pi \cap X$ under π. But as $\Pi \supset \Lambda$, this hyperplane is parameterised by a point on $\Gamma \backslash W_{c,d}$. In particular, Γ is not contained in $W_{c,d}$. This completes the proof. □

The following result is a minor extension of Theorem 2.1 in [**Sal05**].

THEOREM 2.2. *Let $W \subset \mathbb{P}^n$ be a closed subscheme defined over $\mathbb{Q}$ where all irreducible components are of dimension at most two. Let D be the sum of the degrees of all irreducible components of W. Then,*

$$N(W,B) = O_{D,n}(B^3) .$$

If W does not contain any plane spanned by its rational points of height at most B, then

$$N(W,B) = O_{D,n,\varepsilon}(B^{2+\varepsilon}).$$

PROOF. One reduces immediately to the case where W is integral and then to the case where W is geometrically integral by the arguments in the proof of Theorem 2.1 in [**Sal05**]. It is also shown there that Theorem 2.2 holds if W is geometrically integral and not a plane. It remains to prove Theorem 2.2 for a rational plane W. Then the rational points of height $\leq B$ on W span an r-plane Λ, $r \leq 2$ where $N(W,B) = N(\Lambda,B) = O_n(B^{\dim \Lambda})$ if $N(W,B) \geq 1$, ([**HB02**], Lemma 1(iii)). This completes the proof. □

LEMMA 2.3. *Let $X \subset \mathbb{P}^4$ be a geometrically integral projective threefold over $\mathbb{Q}$ of degree $d \geq 2$. Then there are $O_{d,\varepsilon}(B^{11/4+\varepsilon})$ points $x \in \widetilde{S}(X,B)$ for which there is a rational hyperplane $\Pi \subset \mathbb{P}^4$ of height at most $(5B)^{1/4}$ containing x such that $\Pi \cap X$ is not geometrically integral. If X is not a cone over a curve then there are $O_{d,\varepsilon}(B^{5/2+\varepsilon})$ such points $x \in \widetilde{S}(X,B)$.*

PROOF. It suffices to establish this bound under the extra hypothesis that $\Pi \in W_{c,d}(\mathbb{Q})$ for some fixed integer $c < d$. By Lemma 2.1 and Theorem 2.2 we have $N(W_{c,d},(5B)^{1/4}) = O_{d,\varepsilon}(B^{3/4})$ in general and $N(W_{c,d},(5B)^{1/4}) = O_{d,\varepsilon}(B^{1/2+\varepsilon})$ if X is not a cone over a curve. We may also apply Theorem 2.2 to the closure W in $\Pi \cap X$ of the complement of all rational planes in $\Pi \cap X$ spanned by its rational points of height $\leq B$. We then get that there are $O_{d,\varepsilon}(B^{2+\varepsilon})$ points in $\widetilde{S}(X,B) \cap \Pi(\mathbb{Q})$ for any hyperplane $\Pi \subset \mathbb{P}^4$. The desired result follows by summing over all $\Pi \in W_{c,d}(\mathbb{Q})$ in the statement of the lemma and over all c. □

3. The points outside the lines

We shall in this section count the points outside the lines on hypersurfaces X in $\mathbb{P}^4$.

DEFINITION 3.1. A surface $X \subset \mathbb{P}^3$ is said to be a Steiner surface if there is a morphism $\pi : \mathbb{P}^2 \to \mathbb{P}^3$ of projective degree 2 which maps $\mathbb{P}^2$ birationally onto X.

It follows immediately from the definition that a Steiner surface is of degree 4. The following result is proved but not stated in [**Sala**].

THEOREM 3.2. *Let $X \subset \mathbb{P}^3$ be a geometrically integral projective surface over $\mathbb{Q}$ of degree $d \geq 3$. Suppose that X is not a Steiner surface. Then there exists a set of $O_{d,\varepsilon}(B^{3/2\sqrt{d}+\varepsilon})$ rational lines on X such that there are*

$$O_{d,\varepsilon}(B^{3/\sqrt{d}+\varepsilon} + B^{3/2\sqrt{d}+2/3+\varepsilon} + B^{1+\varepsilon})$$

rational points of height $\leq B$ not lying on these lines. If $X \subset \mathbb{P}^3$ is a Steiner surface, then there are

$$O_{d,\varepsilon}(B^{43/28+\varepsilon})$$

rational points of height $\leq B$ not lying on the lines.

PROOF. There exists by Theorem 0.5 in [**Sala**] a set Γ of $O_{d,\varepsilon}(B^{3/2\sqrt{d}+\varepsilon})$ geometrically integral curves of degree $O_{d,\varepsilon}(1)$ on X such that all but $O_{d,\varepsilon}(B^{3/\sqrt{d}+\varepsilon})$ rational points of height $\leq B$ on X lie on the union of these curves. Hence, by [**HB02**], th.5, there are $O_{d,\varepsilon}(B^{3/2\sqrt{d}+2/3+\varepsilon})$ rational points of height $\leq B$ on the