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Singular moduli generating functions for modular curves and surfaces

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Abstract. Zagier [**Zag02**] proved that the generating functions for the traces of singular moduli are often weight 3/2 modular forms. Here we investigate the modularity of generating functions of Maass singular moduli, as well as traces of singular moduli on Hilbert modular surfaces.

1. Introduction and Statement of Results

Let $j(z)$ be the usual modular function for $\mathrm{SL}_2(\mathbb{Z})$

$$j(z) = q^{-1} + 744 + 196884q + 21493760q^2 + \cdots ,$$

where $q = e^{2\pi i z}$. The values of modular functions such as $j(z)$ at imaginary quadratic arguments in $\mathfrak{h}$, the upper half of the complex plane, are known as *singular moduli.* Singular moduli are algebraic integers which play many roles in number theory. For example, they generate class fields of imaginary quadratic fields, and they parameterize isomorphism classes of elliptic curves with complex multiplication.

This expository article describes the author's recent joint works with Bringmann, Bruinier, Jenkins, and Rouse [**BO, BOR05, BJO06**] on generating functions for traces of singular moduli. To motivate these results, we begin by comparing the classical evaluations

$$\frac{j\left(\frac{-1+\sqrt{-3}}{2}\right) - 744}{3} = -248, \qquad \frac{j(i) - 744}{2} = 492, \qquad j\left(\frac{1+\sqrt{-7}}{2}\right) - 744 = -4119,$$

with the Fourier coefficients of the modular form

$$g(z) := -\frac{\eta(z)^2 \cdot E_4(4z)}{\eta(2z)\eta(4z)^6} = -q^{-1} + 2 - 248q^3 + 492q^4 - 4119q^7 + 7256q^8 - \cdots , \tag{1.1}$$

where $E_4(z) = 1 + 240\sum_{n=1}^{\infty} \sigma_3(n)q^n$ is the usual weight 4 Eisenstein series, and $\eta(z) = q^{1/24}\prod_{n=1}^{\infty}(1 - q^n)$ is Dedekind's eta-function. The appearance of singular

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moduli as the initial coefficients of the modular form $g(z)$ is not a coincidence. In a recent groundbreaking paper [**Zag02**], Zagier established that $g(z)$ is indeed the generating function for the "traces" of the $j(z)$ singular moduli. In this important paper, Zagier employs such results to give a new proof of Borcherds' famous theorem on the infinite product expansions of integer weight modular forms on $\mathrm{SL}_2(\mathbb{Z})$ with Heegner divisor (for example, see [**Bor95a, Bor95b**]).

Here we survey three recent papers inspired by Zagier's work. First we revisit his work from the context of Maass-Poincaré series. This uniform approach gives many of his results as special cases of a single theorem, and, as an added bonus, gives exact formulas for traces of singular moduli. Our first general result (see Theorem 1.1) establishes that the coefficients of certain half-integral weight Maass forms have the property that their coefficients are traces of singular moduli. These works are described in [**BO, BJO06**]. Secondly, we obtain generalizations [**BOR05**] for Hilbert modular surfaces (see Theorem 1.2).

Before we state these results, we first recall some of Zagier's results. For integers λ, let $M^!_{\lambda+\frac{1}{2}}$ be the space of weight $\lambda+\frac{1}{2}$ *weakly holomorphic modular forms* on $\Gamma_0(4)$ satisfying the "Kohnen plus-space" condition. Recall that a meromorphic modular form is weakly holomorphic if its poles (if there are any) are supported at the cusps, and it satisfies Kohnen's plus-space condition if its q-expansion has the form

$$\sum_{(-1)^\lambda n\equiv 0,1 \pmod 4} a(n)q^n. \tag{1.2}$$

Throughout, let $d\equiv 0,3 \pmod 4$ be a positive integer, let $H(d)$ be the Hurwitz-Kronecker class number for the discriminant $-d$, and let $\mathcal{Q}_d$ be the set of positive definite integral binary quadratic forms (including imprimitive forms)

$$Q(x,y)=[a,b,c]=ax^2+bxy+cy^2$$

with discriminant $D_Q=-d=b^2-4ac$. For each Q, let τ_Q be the unique root in $\mathfrak{h}$ of $Q(x,1)=0$. The singular modulus $f(\tau_Q)$, for any modular invariant $f(z)$, depends only on the equivalence class of Q under the action of $\Gamma:=\mathrm{PSL}_2(\mathbb{Z})$. If $\omega_Q\in\{1,2,3\}$ is given by

$$\omega_Q:=\begin{cases} 2 & \text{if } Q\sim_\Gamma [a,0,a],\\ 3 & \text{if } Q\sim_\Gamma [a,a,a],\\ 1 & \text{otherwise,}\end{cases}$$

then, for a modular invariant $f(z)$, define the trace $\mathrm{Tr}(f;d)$ by

$$\mathrm{Tr}(f;d):=\sum_{Q\in\mathcal{Q}_d/\Gamma}\frac{f(\tau_Q)}{\omega_Q}. \tag{1.3}$$

Theorems 1 and 5 of [**Zag02**] imply the following.

Theorem. (Zagier)
If $f(z)\in\mathbb{Z}[j(z)]$ has a Fourier expansion with constant term 0, then there is a finite principal part $A_f(z)=\sum_{n\le 0}a_f(n)q^n$ for which

$$A_f(z)+\sum_{0<d\equiv 0,3 \pmod 4}\mathrm{Tr}(f;d)q^d\in M^!_{\frac{3}{2}}.$$

REMARK. The earlier claim about the modular form $g(z)$ is the $f(z) = J_1(z) = j(z) - 744$ case of this theorem.

REMARK. Using Poincaré series constructed [**BJO06**] by Bruinier, Jenkins and the author, Duke [**Duk06**] and Jenkins [**Jen**] have provided new proofs of this theorem by combining earlier results of Niebur [**Nie73**] with facts about Kloosterman-Salié sums.

Zagier gave several generalizations of this result. Here we highlight two of these; the first concerns "twisted traces". For fundamental discriminants D_1, let χ_{D_1} denote the associated genus character for positive definite binary quadratic forms whose discriminants are multiples of D_1. If λ is an integer and D_2 is a nonzero integer for which $(-1)^\lambda D_2 \equiv 0, 1 \pmod 4$ and $(-1)^\lambda D_1 D_2 < 0$, then define the twisted trace of a modular invariant $f(z)$, say $\mathrm{Tr}_{D_1}(f; D_2)$, by

$$\mathrm{Tr}_{D_1}(f; D_2) := \sum_{Q \in \mathcal{Q}_{|D_1 D_2|}/\Gamma} \frac{\chi_{D_1}(Q) f(\tau_Q)}{\omega_Q}. \tag{1.4}$$

If $f \in \mathbb{Z}[j(z)]$ has a Fourier expansion with constant term 0, then Zagier proved that these traces are coefficients of weight 3/2 forms (see Theorem 6 of [**Zag02**]). The second generalization involves $\mathrm{Tr}(f; d)$ for special non-holomorphic modular functions $f(z)$. In these cases, the corresponding generating functions have weight $\lambda + \frac{1}{2}$, where $\lambda \in \{-6, -4, -3, -2, -1\}$ (see Theorems 10 and 11 of [**Zag02**]).

REMARK. Kim [**Kim04, Kim**] has established the modularity for traces of singular moduli on certain genus zero congruence subgroups. Using theta lifts, Bruinier and Funke [**BF06**] (see Theorem 3.1) have recently proven a more general theorem which holds for modular functions on modular curves of arbitrary genus. Their result plays an important role in the proof of Theorem 1.2, our result for Hilbert modular surfaces.

Generalizing the arguments of Duke and Jenkins alluded to above, we show that the coefficients of certain half-integral weight Maass-Poincaré series are traces of singular moduli. This result includes the results of Zagier described above, and, as an added bonus, gives exact formulas for these traces. To construct these series, let $k := \lambda + \frac{1}{2}$, where λ is an arbitrary integer, and let $M_{\nu,\mu}(z)$ be the usual M-Whittaker function. For $s \in \mathbb{C}$ and $y \in \mathbb{R} - \{0\}$, we define

$$\mathcal{M}_s(y) := |y|^{-\frac{k}{2}} M_{\frac{k}{2}\,\mathrm{sgn}(y),\, s-\frac{1}{2}}(|y|).$$

Suppose that $m \geq 1$ is an integer with $(-1)^{\lambda+1} m \equiv 0, 1 \pmod 4$. Define $\varphi_{-m,s}(z)$ by

$$\varphi_{-m,s}(z) := \mathcal{M}_s(-4\pi m y) e(-mx),$$

where $z = x + iy$, and $e(w) := e^{2\pi i w}$. Furthermore, let

$$\Gamma_\infty := \left\{ \pm \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} \;:\; n \in \mathbb{Z} \right\}$$

denote the translations within $\mathrm{SL}_2(\mathbb{Z})$. Using this notation, define the Poincaré series

$$\mathcal{F}_\lambda(-m, s; z) := \sum_{A \in \Gamma_\infty \backslash \Gamma_0(4)} (\varphi_{-m,s} \mid_k A)(z) \tag{1.5}$$

for $\mathrm{Re}(s) > 1$. Here $|_k$ denotes the usual half-integral weight k "slash operator" (see Shimura's seminal paper [**Shi73**]). If pr_λ is Kohnen's projection operator (see page 250 of [**Koh85**]) to the weight $\lambda + \frac{1}{2}$ plus-space for $\Gamma_0(4)$, then for $\lambda \notin \{0, 1\}$ define $F_\lambda(-m; z)$ by

$$F_\lambda(-m;z) := \begin{cases} \frac{3}{2}\mathcal{F}_\lambda\left(-m, \frac{k}{2}; z\right) \mid \mathrm{pr}_\lambda & \text{if } \lambda \geq 2, \\ \frac{3}{2(1-k)\Gamma(1-k)}\mathcal{F}_\lambda\left(-m, 1-\frac{k}{2}; z\right) \mid \mathrm{pr}_\lambda & \text{if } \lambda \leq -1. \end{cases} \tag{1.6}$$

REMARK. For $\lambda = 0$ or 1 we also have series $F_\lambda(-m; z)$. Their construction requires more care. For $\lambda = 1$ this is carried out in [**BJO06**], and for $\lambda = 0$ see [**BO**].

By Theorem 3.5 of [**BJO06**], if $\lambda \geq -6$ with $\lambda \neq -5$, then $F_\lambda(-m; z) \in M^!_{\lambda+\frac{1}{2}}$. For such λ, we denote the corresponding Fourier expansions by

$$F_\lambda(-m;z) = q^{-m} + \sum_{\substack{n\geq 0 \\ (-1)^\lambda n \equiv 0,1 \pmod 4}} b_\lambda(-m;n)q^n \in M^!_{\lambda+\frac{1}{2}}. \tag{1.7}$$

For other λ, namely $\lambda = -5$ or $\lambda \leq -7$, it turns out that the $F_\lambda(-m; z)$ are weak Maass forms of weight $\lambda + \frac{1}{2}$ (see Section 2.1). We denote their expansions by

$$F_\lambda(-m;z) = B_\lambda(-m;z) + q^{-m} + \sum_{\substack{n\geq 0 \\ (-1)^\lambda n \equiv 0,1 \pmod 4}} b_\lambda(-m;n)q^n, \tag{1.8}$$

where $B_\lambda(-m; z)$ is the "non-holomorphic" part of $F_\lambda(-m; z)$.

EXAMPLE. If $\lambda = 1$ and $-m = -1$, then we have the modular form in (1.1)

$$-F_1(-1;z) = g(z) = -q^{-1} + 2 - 248q^3 + 492q^4 - 4119q^7 + 7256q^8 - \cdots.$$

Generalizing Zagier's results, we show that the coefficients $b_\lambda(-m; n)$ of the $F_\lambda(-m; z)$ are traces of singular moduli for functions defined by Niebur [**Nie73**]. If $I_s(x)$ denotes the usual I-Bessel function, and if $\lambda > 1$, then let

$$\mathfrak{F}_\lambda(z) := \pi \sum_{A\in\Gamma_\infty\backslash\mathrm{SL}_2(\mathbb{Z})} \mathrm{Im}(Az)^{\frac{1}{2}} I_{\lambda-\frac{1}{2}}(2\pi\mathrm{Im}(Az))e(-\mathrm{Re}(Az)). \tag{1.9}$$

REMARK. For $\lambda = 1$, Niebur's [**Nie73**] shows that $\mathfrak{F}_1(z) = \frac{1}{2}(j(z) - 744)$, where this function is the analytic continuation, as $s \to 1$ from the right, of

$$-12 + \pi \sum_{A\in\Gamma_\infty\backslash\mathrm{SL}_2(\mathbb{Z})} \mathrm{Im}(Az)^{\frac{1}{2}} I_{s-\frac{1}{2}}(2\pi\mathrm{Im}(Az))e(-\mathrm{Re}(Az)).$$

Arguing as in [**BJO06, Duk06, Jen**], Bringmann and the author have proved [**BO**] the following:

THEOREM 1.1. (Bringmann and Ono; Theorem 1.2 of [**BO**])
If $\lambda, m \geq 1$ are integers for which $(-1)^{\lambda+1}m$ is a fundamental discriminant (which includes 1), then for each positive integer n with $(-1)^\lambda n \equiv 0, 1 \pmod 4$ we have

$$b_\lambda(-m;n) = \frac{2(-1)^{[(\lambda+1)/2]}n^{\frac{\lambda}{2}-\frac{1}{2}}}{m^{\frac{\lambda}{2}}} \cdot \mathrm{Tr}_{(-1)^{\lambda+1}m}(\mathfrak{F}_\lambda; n).$$