

## Negative values of truncations to $L(1, \chi)$

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ABSTRACT. For fixed large  $x$  we give upper and lower bounds for the minimum of  $\sum_{n \leq x} \chi(n)/n$  as we minimize over all real-valued Dirichlet characters  $\chi$ . This follows as a consequence of bounds for  $\sum_{n \leq x} f(n)/n$  but now minimizing over all completely multiplicative, real-valued functions  $f$  for which  $-1 \leq f(n) \leq 1$  for all integers  $n \geq 1$ . Expanding our set to all multiplicative, real-valued multiplicative functions of absolute value  $\leq 1$ , the minimum equals  $-0.4553 \dots + o(1)$ , and in this case we can classify the set of optimal functions.

### 1. Introduction

Dirichlet's celebrated class number formula established that  $L(1, \chi)$  is positive for primitive, quadratic Dirichlet characters  $\chi$ . One might attempt to prove this positivity by trying to establish that the partial sums  $\sum_{n \leq x} \chi(n)/n$  are all non-negative. However, such truncated sums can get negative, a feature which we will explore in this note.

By quadratic reciprocity we may find an arithmetic progression  $(\bmod 4 \prod_{p \leq x} p)$  such that any prime  $q$  lying in this progression satisfies  $\left(\frac{p}{q}\right) = -1$  for each  $p \leq x$ . Such primes  $q$  exist by Dirichlet's theorem on primes in arithmetic progressions, and for such  $q$  we have  $\sum_{n \leq x} \left(\frac{n}{q}\right)/n = \sum_{n \leq x} \lambda(n)/n$  where  $\lambda(n) = (-1)^{\Omega(n)}$  is the Liouville function. Turán [6] suggested that  $\sum_{n \leq x} \lambda(n)/n$  may be always positive, noting that this would imply the truth of the Riemann Hypothesis (and previously Pólya had conjectured that the related  $\sum_{n \leq x} \lambda(n)$  is non-positive for all  $x \geq 2$ , which also implies the Riemann Hypothesis). In [Has58] Haselgrove showed that both the Turán and Pólya conjectures are false (in fact  $x = 72, 185, 376, 951, 205$  is the smallest integer  $x$  for which  $\sum_{n \leq x} \lambda(n)/n < 0$ , as was recently determined in [BFM]). We therefore know that truncations to  $L(1, \chi)$  may get negative.

Let  $\mathcal{F}$  denote the set of all completely multiplicative functions  $f(\cdot)$  with  $-1 \leq f(n) \leq 1$  for all positive integers  $n$ , let  $\mathcal{F}_1$  be those for which each  $f(n) = \pm 1$ , and  $\mathcal{F}_0$  be those for which each  $f(n) = 0$  or  $\pm 1$ . Given any  $x$  and any  $f \in \mathcal{F}_0$  we may find a primitive quadratic character  $\chi$  with  $\chi(n) = f(n)$  for all  $n \leq x$  (again, by using

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2000 *Mathematics Subject Classification*. Primary 11M20.

Le premier auteur est partiellement soutenu par une bourse du Conseil de recherches en sciences naturelles et en génie du Canada. The second author is partially supported by the National Science Foundation and the American Institute of Mathematics (AIM).

quadratic reciprocity and Dirichlet's theorem on primes in arithmetic progressions) so that, for any  $x \geq 1$ ,

$$\min_{\chi \text{ a quadratic character}} \sum_{n \leq x} \frac{\chi(n)}{n} = \delta_0(x) := \min_{f \in \mathcal{F}_0} \sum_{n \leq x} \frac{f(n)}{n}.$$

Moreover, since  $\mathcal{F}_1 \subset \mathcal{F}_0 \subset \mathcal{F}$  we have that

$$\delta(x) := \min_{f \in \mathcal{F}} \sum_{n \leq x} \frac{f(n)}{n} \leq \delta_0(x) \leq \delta_1(x) := \min_{f \in \mathcal{F}_1} \sum_{n \leq x} \frac{f(n)}{n}.$$

We expect that  $\delta(x) \sim \delta_1(x)$  and even, perhaps, that  $\delta(x) = \delta_1(x)$  for sufficiently large  $x$ .

Trivially  $\delta(x) \geq -\sum_{n \leq x} 1/n = -(\log x + \gamma + O(1/x))$ . Less trivially  $\delta(x) \geq -1$ , as may be shown by considering the non-negative multiplicative function  $g(n) = \sum_{d|n} f(d)$  and noting that

$$0 \leq \sum_{n \leq x} g(n) = \sum_{d \leq x} f(d) \left[ \frac{x}{d} \right] \leq \sum_{d \leq x} \left( x \frac{f(d)}{d} + 1 \right).$$

We will show that  $\delta(x) \leq \delta_1(x) < 0$  for all large values of  $x$ , and that  $\delta(x) \rightarrow 0$  as  $x \rightarrow \infty$ .

**THEOREM 1.** *For all large  $x$  and all  $f \in \mathcal{F}$  we have*

$$\sum_{n \leq x} \frac{f(n)}{n} \geq -\frac{1}{(\log \log x)^{\frac{3}{5}}}.$$

*Further, there exists a constant  $c > 0$  such that for all large  $x$  there exists a function  $f(= f_x) \in \mathcal{F}_1$  such that*

$$\sum_{n \leq x} \frac{f(n)}{n} \leq -\frac{c}{\log x}.$$

*In other words, for all large  $x$ ,*

$$-\frac{1}{(\log \log x)^{\frac{3}{5}}} \leq \delta(x) \leq \delta_0(x) \leq \delta_1(x) \leq -\frac{c}{\log x}.$$

Note that Theorem 1 implies that there exists some absolute constant  $c_0 > 0$  such that  $\sum_{n \leq x} f(n)/n \geq -c_0$  for all  $x$  and all  $f \in \mathcal{F}$ , and that equality occurs only for bounded  $x$ . It would be interesting to determine  $c_0$  and all  $x$  and  $f$  attaining this value, which is a feasible goal developing the methods of this article.

It would be interesting to determine more precisely the asymptotic nature of  $\delta(x)$ ,  $\delta_0(x)$  and  $\delta_1(x)$ , and to understand the nature of the optimal functions.

Instead of completely multiplicative functions we may consider the larger class  $\mathcal{F}^*$  of multiplicative functions, and analogously define

$$\delta^*(x) := \min_{f \in \mathcal{F}^*} \sum_{n \leq x} \frac{f(n)}{n}.$$

**THEOREM 2.** *We have*

$$\delta^*(x) = \left( 1 - 2 \log(1 + \sqrt{e}) + 4 \int_1^{\sqrt{e}} \frac{\log t}{t+1} dt \right) \log 2 + o(1) = -0.4553 \dots + o(1).$$

If  $f^* \in \mathcal{F}^*$  and  $x$  is large then

$$\sum_{n \leq x} \frac{f^*(n)}{n} \geq -\frac{1}{(\log \log x)^{\frac{3}{5}}},$$

unless

$$\sum_{k=1}^{\infty} \frac{1 + f^*(2^k)}{2^k} \ll (\log x)^{-\frac{1}{20}}.$$

Finally

$$\sum_{n \leq x} \frac{f^*(n)}{n} = \delta^*(x) + o(1)$$

if and only if

$$\left( \sum_{k=1}^{\infty} \frac{1 + f^*(2^k)}{2^k} \right) \log x + \sum_{3 \leq p \leq x^{1/(1+\sqrt{\epsilon})}} \sum_{k=1}^{\infty} \frac{1 - f^*(p^k)}{p^k} + \sum_{x^{1/(1+\sqrt{\epsilon})} \leq p \leq x} \frac{1 + f^*(p)}{p} = o(1).$$

## 2. Constructing negative values

Recall Haselgrove's result [Has58]: there exists an integer  $N$  such that

$$\sum_{n \leq N} \frac{\lambda(n)}{n} = -\delta$$

with  $\delta > 0$ , where  $\lambda \in \mathcal{F}_1$  with  $\lambda(p) = -1$  for all primes  $p$ . Let  $x > N^2$  be large and consider the function  $f = f_x \in \mathcal{F}_1$  defined by  $f(p) = 1$  if  $x/(N+1) < p \leq x/N$  and  $f(p) = -1$  for all other  $p$ . If  $n \leq x$  then we see that  $f(n) = \lambda(n)$  unless  $n = p\ell$  for a (unique) prime  $p \in (x/(N+1), x/N]$  in which case  $f(n) = \lambda(\ell) = \lambda(n) + 2\lambda(\ell)$ . Thus

$$\begin{aligned} \sum_{n \leq x} \frac{f(n)}{n} &= \sum_{n \leq x} \frac{\lambda(n)}{n} + 2 \sum_{x/(N+1) < p \leq x/N} \frac{1}{p} \sum_{\ell \leq x/p} \frac{\lambda(\ell)}{\ell} \\ (2.1) \qquad &= \sum_{n \leq x} \frac{\lambda(n)}{n} - 2\delta \sum_{x/(N+1) < p \leq x/N} \frac{1}{p}. \end{aligned}$$

A standard argument, as in the proof of the prime number theorem, shows that

$$\sum_{n \leq x} \frac{\lambda(n)}{n} = \frac{1}{2\pi i} \int_{2-i\infty}^{2+i\infty} \frac{\zeta(2s+2)}{\zeta(s+1)} \frac{x^s}{s} ds \ll \exp(-c\sqrt{\log x}),$$

for some  $c > 0$ . Further, the prime number theorem readily gives that

$$\sum_{x/(N+1) < p \leq x/N} \frac{1}{p} \sim \log \left( \frac{\log(x/N)}{\log(x/(N+1))} \right) \asymp \frac{1}{N \log x}.$$

Inserting these estimates in (2.1) we obtain that  $\delta(x) \leq -c/\log x$  for large  $x$  (here  $c \asymp \delta/N$ ), as claimed in Theorem 1.

REMARK 2.1. In [BFM] it is shown that one can take  $\delta = 2.0757641 \dots \cdot 10^{-9}$  for  $N = 72204113780255$  and therefore we may take  $c \approx 2.87 \cdot 10^{-23}$ .

### 3. The lower bound for $\delta(x)$

PROPOSITION 3.1. *Let  $f$  be a completely multiplicative function with  $-1 \leq f(n) \leq 1$  for all  $n$ , and set  $g(n) = \sum_{d|n} f(d)$  so that  $g$  is a non-negative multiplicative function. Then*

$$\sum_{n \leq x} \frac{f(n)}{n} = \frac{1}{x} \sum_{n \leq x} g(n) + (1 - \gamma) \frac{1}{x} \sum_{n \leq x} f(n) + O\left(\frac{1}{(\log x)^{\frac{1}{5}}}\right).$$

PROOF. Define  $F(t) = \frac{1}{t} \sum_{n \leq t} f(n)$ . We will make use of the fact that  $F(t)$  varies slowly with  $t$ . From [GS03, Corollary 3], we find that if  $1 \leq w \leq x/10$  then

$$(3.1) \quad \left| |F(x)| - |F(x/w)| \right| \ll \left(\frac{\log 2w}{\log x}\right)^{1-\frac{2}{\pi}} \log\left(\frac{\log x}{\log 2w}\right) + \frac{\log \log x}{(\log x)^{2-\sqrt{3}}}.$$

We may easily deduce that

$$(3.2) \quad \left| F(x) - F(x/w) \right| \ll \left(\frac{\log 2w}{\log x}\right)^{1-\frac{2}{\pi}} \log\left(\frac{\log x}{\log 2w}\right) + \frac{\log \log x}{(\log x)^{2-\sqrt{3}}} \ll \left(\frac{\log 2w}{\log x}\right)^{\frac{1}{4}}.$$

Indeed, if  $F(x)$  and  $F(x/w)$  are of the same sign then (3.2) follows at once from (3.1). If  $F(x)$  and  $F(x/w)$  are of opposite signs then we may find  $1 \leq v \leq w$  with  $|\sum_{n \leq x/v} f(n)| \leq 1$  and then using (3.1) first with  $F(x)$  and  $F(x/v)$ , and second with  $F(x/v)$  and  $F(x/w)$  we obtain (3.2).

We now turn to the proof of the Proposition. We start with

$$(3.3) \quad \sum_{n \leq x} g(n) = \sum_{d \leq x} f(d) \left[ \frac{x}{d} \right] = x \sum_{d \leq x} \frac{f(d)}{d} - \sum_{d \leq x} f(d) \left\{ \frac{x}{d} \right\}.$$

Now

$$\begin{aligned} \sum_{d \leq x} f(d) \left\{ \frac{x}{d} \right\} &= \sum_{j \leq x} \sum_{x/(j+1) < d \leq x/j} f(d) \left( \frac{x}{d} - j \right) \\ &= \sum_{j \leq \log x} \int_{x/(j+1)}^{x/j} \frac{x}{t^2} \sum_{x/(j+1) < d \leq t} f(d) dt + O\left(\frac{x}{\log x}\right). \end{aligned}$$

From (3.2) we see that if  $j \leq \log x$ , and  $x/(j+1) < t \leq x/j$  then

$$\sum_{x/(j+1) < d \leq t} f(d) = \left( t - \frac{x}{(j+1)} \right) \frac{1}{x} \sum_{n \leq x} f(n) + O\left(\frac{x \log(j+1)}{j(\log x)^{\frac{1}{4}}}\right).$$

Using this above we conclude that

$$(3.4) \quad \sum_{d \leq x} f(d) \left\{ \frac{x}{d} \right\} = \left( \sum_{n \leq x} f(n) \right) \sum_{j \leq \log x} \left( \log\left(\frac{j+1}{j}\right) - \frac{1}{j+1} \right) + O\left(\frac{x(\log \log x)^2}{(\log x)^{\frac{1}{4}}}\right).$$

Since  $\sum_{j \leq J} (\log(1+1/j) - 1/(j+1)) = \log(J+1) - \sum_{j \leq J+1} 1/j+1 = 1 - \gamma + O(1/J)$ , when we insert (3.4) into (3.3) we obtain the Proposition.  $\square$