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Second moments of GL_2 automorphic L-functions

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Abstract. The main objective of this paper is to explore a variant of the Rankin-Selberg method introduced by Anton Good about twenty years ago in the context of second integral moments of L-functions attached to modular forms on $SL_2(\mathbb{Z})$. By combining Good's idea with some novel techniques, we shall establish the meromorphic continuation and sharp polynomial growth estimates for certain functions of two complex variables (double Dirichlet series) naturally attached to second integral moments.

1. Introduction

In 1801, in the Disquisitiones Arithmeticae [**Gau01**], Gauss introduced the class number $h(d)$ as the number of inequivalent binary quadratic forms of discriminant d. Gauss conjectured that the average value of $h(d)$ is $\frac{2\pi}{7\zeta(3)}\sqrt{|d|}$ for negative discriminants d. This conjecture was first proved by I. M. Vinogradov [**Vin18**] in 1918. Remarkably, Gauss also made a similar conjecture for the average value of $h(d)\log(\epsilon_d)$, where d ranges over positive discriminants and ϵ_d is the fundamental unit of the real quadratic field $\mathbb{Q}(\sqrt{d})$. Of course, Gauss did not know what a fundamental unit of a real quadratic field was, but he gave the definition that $\epsilon_d = \frac{t+u\sqrt{d}}{2}$, where t, u are the smallest positive integral solutions to Pell's equation $t^2 - du^2 = 4$. For example, he conjectured that

$$d \equiv 0 \pmod 4 \to \sum_{d\le x} h(d)\log(\epsilon_d) \;\sim\; \frac{4\pi^2}{21\zeta(3)} x^{\frac{3}{2}},$$

while

$$d \equiv 1 \pmod 4 \to \sum_{d\le x} h(d)\log(\epsilon_d) \;\sim\; \frac{\pi^2}{18\zeta(3)} x^{\frac{3}{2}}.$$

These latter conjectures were first proved by C. L. Siegel [**Sie44**] in 1944.

In 1831, Dirichlet introduced his famous L–functions

$$L(s,\chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s},$$

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where χ is a character (mod q) and $\Re(s) > 1$. The study of moments

$$\sum_q L(s, \chi_q)^m,$$

say, where χ_q is the real character associated to a quadratic field $\mathbb{Q}(\sqrt{q})$, was not achieved until modern times. In the special case when $s = 1$ and $m = 1$, the value of the first moment reduces to the aforementioned conjecture of Gauss because of the Dirichlet class number formula (see [**Dav00**], pp. 43-53) which relates the special value of the L–function $L(1, \chi_q)$ with the class number and fundamental unit of the quadratic field $\mathbb{Q}(\sqrt{q})$.

Let

$$L(s) = \sum_{n=1}^{\infty} a(n) n^{-s}$$

be the L–function associated to a modular form for the modular group. The main focus of this paper is to obtain meromorphic continuation and growth estimates in the complex variable w of the Dirichlet series

$$\int_1^{\infty} \left|L\left(\tfrac{1}{2} + it\right)\right|^k t^{-w} dt.$$

We shall show, by a new method, that it is possible to obtain meromorphic continuation and rather strong growth estimates of the above Dirichlet series for the case $k = 2$. It is then possible, by standard methods, to obtain asymptotics, as $T \to \infty$, for the second integral moment

$$\int_0^T \left|L(\tfrac{1}{2} + it)\right|^2 dt.$$

In the special case that the modular form is an Eisenstein series this yields asymptotics for the fourth moment of the Riemann zeta-function.

Moment problems associated to the Riemann zeta-function $\zeta(s) = \sum_{n=1}^{\infty} n^{-s}$ were intensively studied in the beginning of the last century. In 1918, Hardy and Littlewood [**HL18**] obtained the second moment

$$\int_0^T \left|\zeta\left(\tfrac{1}{2} + it\right)\right|^2 dt \sim T \log T,$$

and in 1926, Ingham [**Ing26**], obtained the fourth moment

$$\int_0^T \left|\zeta\left(\tfrac{1}{2} + it\right)\right|^4 dt \sim \frac{1}{2\pi^2} \cdot T(\log T)^4.$$

Heath-Brown (1979) [**HB81**] obtained the fourth moment with error term of the form

$$\int_0^T \left|\zeta\left(\tfrac{1}{2} + it\right)\right|^4 dt = \frac{1}{2\pi^2} \cdot T \cdot P_4(\log T) + \mathcal{O}\left(T^{\frac{7}{8}+\epsilon}\right),$$

where $P_4(x)$ is a certain polynomial of degree four.

Let $f(z) = \sum_{n=1}^{\infty} a(n) e^{2\pi i n z}$ be a cusp form of weight κ for the modular group with associated L–function $L_f(s) = \sum_{n=1}^{\infty} a(n) n^{-s}$. Anton Good [**Goo82**] made a significant breakthrough in 1982 when he proved that

$$\int_0^T |L_f\left(\tfrac{\kappa}{2}+it\right)|^2\,dt \;=\; 2aT(\log(T)+b)+\mathcal{O}\left(\left(T\log T\right)^{\frac{2}{3}}\right)$$

for certain constants a, b. It seems likely that Good's method can apply to Eisenstein series.

In 1989, Zavorotny [**Zav89**], improved Heath-Brown's 1979 error term to

$$\int_0^T \left|\zeta\left(\tfrac{1}{2}+it\right)\right|^4\,dt \;=\; \frac{1}{2\pi^2}\cdot T\cdot P_4(\log T) \;+\; \mathcal{O}\left(T^{\frac{2}{3}+\epsilon}\right).$$

Shortly afterwards, Motohashi [**Mot92**], [**Mot93**] slightly improved the above error term to

$$\mathcal{O}\left(T^{\frac{2}{3}}(\log T)^B\right)$$

for some constant $B>0$. Motohashi introduced the double Dirichlet series [**Mot95**], [**Mot97**]

$$\int_1^\infty \zeta(s+it)^2\zeta(s-it)^2 t^{-w}\,dt$$

into the picture and gave a spectral interpretation to the moment problem.

Unfortunately, an old paper of Anton Good [**Goo86**], going back to 1985, which had much earlier outlined an alternative approach to the second moment problem for $GL(2)$ automorphic forms using Poincaré series has been largely forgotten. Using Good's approach, it is possible to recover the aforementioned results of Zavorotny and Motohashi. It is also possible to generalize this method to more general situations; for instance see [**DG**], where the case of $GL(2)$ automorphic forms over an imaginary quadratic field is considered. Our aim here is to explore Good's method and show that it is, in fact, an exceptionally powerful tool for the study of moment problems.

Second moments of $GL(2)$ Maass forms were investigated in [**Jut97**], [**Jut05**]. Higher moments of L–functions associated to automorphic forms seem out of reach at present. Even the conjectured values of such moments were not obtained until fairly recently (see [**CF00**], [**CG01**], [**CFK$^+$**], [**CG84**], [**DGH03**], [**KS99**], [**KS00**]).

Let $\mathcal{H}$ denote the upper half-plane. A complex valued function f defined on $\mathcal{H}$ is called an automorphic form for $\Gamma = SL_2(\mathbb{Z})$, if it satisfies the following properties:

(1) We have

$$f\left(\frac{az+b}{cz+d}\right) \;=\; (cz+d)^\kappa f(z) \qquad \text{for } \begin{pmatrix} a & b \\ c & d \end{pmatrix}\in\Gamma;$$

(2) $f(iy)=\mathcal{O}(y^\alpha)$ for some α, as $y\to\infty$;

(3) κ is either an even positive integer and f is holomorphic, or $\kappa = 0$, in which case, f is an eigenfunction of the non-euclidean Laplacian $\Delta = -y^2\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)$ $(z = x+iy\in\mathcal{H})$ with eigenvalue λ. In the first case, we call f a modular form of weight κ, and in the second, we call f a Maass form with eigenvalue λ.

In addition, if f satisfies

$$\int_0^1 f(x+iy)\,dx \;=\; 0,$$

then it is called a cusp form.

Let f and g be two cusp forms for Γ of the same weight κ (for Maass forms we take $\kappa = 0$) with Fourier expansions

$$f(z) \;=\; \sum_{m\neq 0} a_m\, |m|^{\frac{\kappa-1}{2}}\, W(mz),\;\; g(z) \;=\; \sum_{n\neq 0} b_n\, |n|^{\frac{\kappa-1}{2}}\, W(nz)\;\; (z = x+iy,\; y>0).$$

Here, if f, for example, is a modular form, $W(z) = e^{2\pi i z}$, and the sum is restricted to $m \geq 1$, while if f is a Maass form with eigenvalue $\lambda_1 = \frac{1}{4} + r_1^2$,

$$W(z) \;=\; W_{\frac{1}{2}+ir_1}(z) \;=\; y^{\frac{1}{2}} K_{ir_1}(2\pi y) e^{2\pi i x} \qquad (z = x+iy,\; y>0),$$

where $K_\nu(y)$ is the K–Bessel function. Throughout, we shall assume that both f and g are eigenfunctions of the Hecke operators, normalized so that the first Fourier coefficients $a_1 = b_1 = 1$. Furthermore, if f and g are Maass cusp forms, we shall assume them to be even.

Associated to f and g, we have the L–functions:

$$L_f(s) \;=\; \sum_{m=1}^{\infty} a_m m^{-s}; \qquad L_g(s) \;=\; \sum_{n=1}^{\infty} b_n n^{-s}.$$

In [**Goo86**], Anton Good found a natural method to obtain the meromorphic continuation of multiple Dirichlet series of type

$$\int_1^\infty L_f(s_1+it) L_g(s_2-it)\, t^{-w}\, dt, \tag{1.1}$$

where $L_f(s)$ and $L_g(s)$ are the L–functions associated to automorphic forms f and g on $GL(2,\mathbb{Q})$. For fixed g and fixed $s_1, s_2, w \in \mathbb{C}$, the integral (1.1) may be interpreted as the image of a linear map from the Hilbert space of cusp forms to $\mathbb{C}$ given by

$$f \;\longrightarrow\; \int_1^\infty L_f(s_1+it) L_g(s_2-it)\, t^{-w}\, dt.$$

The Riesz representation theorem guarantees that this linear map has a kernel. Good computes this kernel explicitly. For example when $s_1 = s_2 = \frac{1}{2}$, he shows that there exists a Poincaré series P_w and a certain function K such that

$$\langle f,\; \bar{P}_w g\rangle = \int_{-\infty}^{\infty} L_f(\tfrac{1}{2}+it)\overline{L_g(\tfrac{1}{2}+it)}\; K(t,w)\, dt,$$

where $\langle\ ,\ \rangle$ denotes the Petersson inner product on the Hilbert space of cusp forms. Remarkably, it is possible to choose P_w so that

$$K(t,w) \sim |t|^{-w}, \qquad (\text{as } |t| \to \infty).$$

Good's approach has been worked out for congruence subgroups in [**Zha**].

There are, however, two serious obstacles in Good's method.

- *Although $K(t,w) \sim |t|^{-w}$ as $|t| \to \infty$ and w fixed, it has a quite different behavior when $t \ll |\Im(w)|$. In this case it grows exponentially in $|t|$.*
- *The function $\langle f,\; \bar{P}_w g\rangle$ has infinitely many poles in w, occurring at the eigenvalues of the Laplacian. So there is a problem to obtain polynomial growth in w by the use of convexity estimates such as the Phragmén-Lindelöf theorem.*