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CM points and weight 3/2 modular forms

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Abstract. We survey the results of [**Fun02**] and of our joint work with Bruinier [**BF06**] on using the theta correspondence for the dual pair $\mathrm{SL}(2)\times\mathrm{O}(1,2)$ to realize generating series of values of modular functions on a modular curve as (non)-holomorphic modular forms of weight 3/2.

1. Introduction

The theta correspondence has been an important tool in the theory of automorphic forms with manifold applications to arithmetic questions.

In this paper, we consider a specific theta lift for an isotropic quadratic space V over $\mathbb{Q}$ of signature $(1,2)$. The theta kernel we employ associated to the lift has been constructed by Kudla-Millson (e.g., [**KM86, KM90**]) in much greater generality for $\mathrm{O}(p,q)$ ($\mathrm{U}(p,q)$) to realize generating series of cohomological intersection numbers of certain, 'special' cycles in locally symmetric spaces of orthogonal (unitary) type as holomorphic Siegel (Hermitian) modular forms. In our case for $\mathrm{O}(1,2)$, the underlying locally symmetric space M is a modular curve, and the special cycles, parametrized by positive integers N, are the classical CM points $Z(N)$; i.e., quadratic irrationalities of discriminant $-N$ in the upper half plane.

We survey the results of [**Fun02**] and of our joint work with Bruinier [**BF06**] on using this particular theta kernel to define lifts of various kinds of functions F on the underlying modular curve M. The theta lift is given by

$$I(\tau,F)=\int_M F(z)\theta(\tau,z), \tag{1.1}$$

where $\tau\in\mathbb{H}$, the upper half plane, $z\in M$, and $\theta(\tau,z)$ is the theta kernel in question. Then $I(\tau,F)$ is a (in general non-holomorphic) modular form of weight 3/2 for a congruence subgroup of $\mathrm{SL}_2(\mathbb{Z})$. One key feature of the theta kernel is its very rapid decay on M, which distinguishes it from other theta kernels which are usually moderately increasing. Consequently, we can lift some rather nonstandard, even exponentially increasing, functions F.

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Note that Kudla and Millson, who focus entirely on the (co)homological aspects of their general lift, study in this situation only the lift of the constant function 1 in the compact case of a Shimura curve, when V is anisotropic.

One feature of our work is that it provides a uniform approach to several topics and (in part previously known) results, which so far all have been approached by (entirely) different methods. We discuss the following cases in some detail:

(i) The lift of the constant function 1. Then $I(\tau, 1)$ realizes the generating series of the (geometric) degree of the 0-cycles $Z(N)$ as the holomorphic part of a non-holomorphic modular form. As a special case, we recover Zagier's well known Eisenstein series $\mathcal{F}(\tau, s)$ of weight $3/2$ at $s = 1/2$ (in our normalization) whose Fourier coefficients of positive index are given by the Kronecker-Hurwitz class numbers $H(N)$ [**Zag75, HZ76**].

(ii) The lift of a modular function f of weight 0 on M. In that case, we obtain a generalization with a completely different proof of Zagier's influential result [**Zag02**] on the generating series of the traces of the singular moduli, that is, the sum of values of the classical j-invariant over the CM points of a given discriminant. Moreover, our method provides a generalization to modular curves of arbitrary genus.

(iii) The lift of the logarithm of the Petersson metric $\log \|\Delta\|$ of the discriminant function Δ. This was suggested to us by U. Kühn. In that case, the lift $I(\tau, \|\log \Delta\|)$ turns out to be the *derivative* of Zagier's Eisenstein series $\mathcal{F}'(\tau, s)$ at $s = 1/2$. Furthermore, one can interpret the Fourier coefficients as the *arithmetic* degree of the ($\mathbb{Z}$ extension of the) CM cycles. This provides a different approach for the result of (Kudla, Rapoport and) Yang [**Yan04**] in this case, part of Kudla's general program on realizing generating series in arithmetic geometry as modular forms, in particular as derivatives of Eisenstein series. Their result in the modular curve case grew out of their extensive and deep work on the analogous but more involved case for Shimura curves [**KRY04, KRY06**].

(iv) The lift of a weight 0 Maass cusp form f on M. For this input, our lift is equivalent to a theta lift introduced by Maass [**Maa59**], which was studied and applied by Duke [**Duk88**] (to obtain equidistribution results for the CM points and certain geodesics in M) and Katok and Sarnak [**KS93**] (to obtain nonnegativity of the L-function of f at the center of the criticial strip).

The paper is mostly expository; for convenience of the reader and for future use, we briefly discuss the construction of the theta kernel and also give general formulas for the Fourier coefficients.

However, we also discuss a few new aspects. Namely:

(v) For any meromorphic modular form f, we give an explicit formula for the positive Fourier coefficients of the lift $I(\tau, \log \|f\|)$ of the logarithm of the Petersson metric of f in the case when the divisor of f is *not* (necessarily) disjoint to one of the 0-cycles $Z(N)$. In particular, for the j-invariant, we realize the logarithm of the *norm* of the singular moduli as the Fourier coefficients of a non-holomorphic modular form of weight $3/2$. Recall that the norms of the singular moduli were studied by Gross-Zagier [**GZ85**].

In this context and also in view of (iii) it will be interesting to consider the lift for the logarithm of the Petersson metric of a Borcherds product [**Bor98**]. We will come back to this point in the near future.

(vi) Bringmann, Ono, and Rouse [**BOR05**] consider the intersection of a modular curve with a Hirzebruch-Zagier curve T_N in a Hilbert modular curve. Based on our work, they realize the generating series of the traces of the singular moduli on these intersections as a weakly holomorphic modular form of weight 2. They proceed to find some beautiful formulas involving Hilbert class polynomials.

In the last section of this paper, we show how one can obtain such generating series in the context of the Kudla-Millson machinery and generalize this aspect of [**BOR05**] to the intersection of a modular curve with certain special divisors inside locally symmetric spaces associated to $\mathrm{O}(n, 2)$.

Some comments on the usage of this particular kernel function for the lift are in order. The lift I is designed to produce *holomorphic* generating series, while often theta series and integrals associated to indefinite quadratic forms give rise to non-holomorphic modular forms. Furthermore, the lift focuses a priori only on the positive coefficients which correspond to the CM points, while the negative coefficients (which correspond to certain geodesics in M) often vanish. For these geodesics, in the Kudla-Millson theory [**KM86, KM90**], there is another lift for signature $(2, 1)$ with weight 2 forms as input, which produces generating series of periods over the geodesics, see also [**FM02**]. This lift is closely related to Shintani's theta lift [**Shi75**].

Finally note that J. Bruinier [**Bru06**] wrote up a survey on some aspects of our work as well. I also thank him and U. Kühn for comments on the present paper. We also thank the Centre de Recerca Matemàtica in Bellaterra/Spain for its hospitality during fall 2005.

2. Basic notions

2.1. CM points. Let V be a rational vector space of dimension 3 with a non-degenerate symmetric bilinear form $(\, , \,)$ of signature $(1, 2)$. We assume that V is given by

$$V = \{X \in M_2(\mathbb{Q}); \mathrm{tr}(X) = 0\} \tag{2.1}$$

with $(X, Y) = \mathrm{tr}(XY)$ and associated quadratic form $q(X) = \frac{1}{2}(X, X) = \det(X)$. We let $\underline{G} = \mathrm{Spin}\, V \simeq \mathrm{SL}_2$, which acts on V by $g.X := gXg^{-1}$. We set $G = \underline{G}(\mathbb{R})$ and let $D = G/K$ be the associated symmetric space, where $K = \mathrm{SO}(2)$ is the standard maximal compact subgroup of G. We have $D \simeq \mathbb{H} = \{z \in \mathbb{C};\ \Im(z) > 0\}$. Let $L \subset V(\mathbb{Q})$ be an integral lattice of full rank and let Γ be a congruence subgroup of G which takes L to itself. We write $M = \Gamma\backslash D$ for the attached locally symmetric space, which is a modular curve. Throughout the paper let p be a prime or $p = 1$. For simplicity, we assume that the lattice L is given by

$$L = \left\{ [a, b, c] := \begin{pmatrix} b & -2c \\ 2ap & -b \end{pmatrix} :\ a, b, c \in \mathbb{Z} \right\}. \tag{2.2}$$

(For arbitrary even lattices, see [**BF06**]). Then we can take $\Gamma = \Gamma_0^*(p)$, the extension of the Hecke subgroup $\Gamma_0(p)$ by the Fricke involution W_p. Note that then M has only one cusp.

We identify D with the space of lines in $V(\mathbb{R})$ on which the form $(\ ,\ )$ is positive:

$$D \simeq \{z \subset V(\mathbb{R});\ \dim z = 1 \text{ and } (\ ,\ )|_z > 0\}. \tag{2.3}$$

We pick as base point of D the line z_0 spanned by $\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$. For $z \in \mathbb{H}$, we choose $g_z \in G/K$ such that $g_z i = z$; the action is the usual linear fractional transformation on $\mathbb{H}$. Then $z \longmapsto g_z z_0$ gives rise to a G-equivariant isomorphism $\mathbb{H} \simeq D$. The positive line associated to $z = x + iy \in \mathbb{H}$ is generated by $X(z) := g_z . \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$. We let $(\ ,\ )_z$ be the minimal majorant of $(\ ,\ )$ associated to $z \in D$. One easily sees that $(X,X)_z = (X, X(z))^2 - (X,X)$.

The classical CM points are now given as follows. For $X = [a,b,c] \in V$ such that $q(X) = 4acp - b^2 = N > 0$, we put

$$D_X = \operatorname{span}(X) \in D. \tag{2.4}$$

It is easy to see that D_X is explicitly given by the point $\frac{-b+i\sqrt{N}}{2ap}$ in the upper half plane. The stabilizer Γ_X of X in Γ is finite. We then denote by $Z(X)$ the image of D_X in M, counted with multiplicity $\frac{1}{|\overline{\Gamma}_X|}$. Here $\overline{\Gamma}_X$ denotes the image of Γ_X in $\mathrm{PGL}_2(\mathbb{Z})$. Furthermore, Γ acts on $L_N = \{X \in L;\ q(X) = N\}$ with finitely many orbits. The CM points of discriminant $-N$ are given by

$$Z(N) = \sum_{X \in \Gamma \backslash L_N} Z(X). \tag{2.5}$$

We can interpret this in terms of positive definite binary quadratic forms as well. For $N > 0$ a positive integer, we let $\mathcal{Q}_{N,p}$ be the set of positive definite binary quadratic forms of the form $apX^2 + bXY + cY^2$ of discriminant $-N = b^2 - 4acp$ with $a, b, c, \in \mathbb{Z}$. Then $\Gamma = \Gamma_0^*(p)$ acts on $\mathcal{Q}_{N,p}$ in the usual way, and the obvious map from $\mathcal{Q}_{N,p}$ to L_N is $\Gamma_0^*(p)$-equivariant, and L_N is in bijection with $\mathcal{Q}_{N,p} \coprod -\mathcal{Q}_{N,p}$. (The vector $X = [a,b,c] \in L_N$ with $a < 0$ corresponds to a negative definite form).

For a Γ-invariant function F on $D \simeq \mathbb{H}$, we define its trace by

$$\mathbf{t}_F(N) = \sum_{z \in Z(N)} F(z) = \sum_{X \in \Gamma \backslash L_N} \frac{1}{|\overline{\Gamma}_X|} F(D_X). \tag{2.6}$$

2.2. The Theta lift. Kudla and Millson [**KM86**] have explicitly constructed a Schwartz function $\varphi_{KM} = \varphi$ on $V(\mathbb{R})$ valued in $\Omega^{1,1}(D)$, the differential $(1,1)$-forms on D. It is given by

$$\varphi(X,z) = \left((X, X(z))^2 - \frac{1}{2\pi} \right) e^{-\pi (X,X)_z}\, \omega, \tag{2.7}$$

where $\omega = \frac{dx \wedge dy}{y^2} = \frac{i}{2} \frac{dz \wedge d\bar{z}}{y^2}$. We have $\varphi(g.X, gz) = \varphi(X,z)$ for $g \in G$. We define

$$\varphi^0(X,z) = e^{\pi(X,X)} \varphi(X,z) = \left((X, X(z))^2 - \frac{1}{2\pi} \right) e^{-2\pi R(X,z)}\, \omega, \tag{2.8}$$

with $R(X,z) = \frac{1}{2}(X,X)_z - \frac{1}{2}(X,X)$. Note tha $R(X,z) = 0$ if and only if $z = D_X$, i.e., if X lies in the line generated by $X(z)$.

For $\tau = u + iv \in \mathbb{H}$, we put $g'_\tau = \left(\begin{smallmatrix} 1 & u \\ 0 & 1 \end{smallmatrix}\right) \begin{pmatrix} v^{1/2} & 0 \\ 0 & v^{-1/2} \end{pmatrix}$, and we define

$$\varphi(X,\tau,z) = \varphi^0(\sqrt{v}X, z) e^{2\pi i q(X)\tau}. \tag{2.9}$$