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An overview of Manin's conjecture for del Pezzo surfaces

T.D. Browning

Abstract. This paper surveys recent progress towards the Manin conjecture for (singular and non-singular) del Pezzo surfaces. To illustrate some of the techniques available, an upper bound of the expected order of magnitude is established for a singular del Pezzo surface of degree four.

1. Introduction

A fundamental theme in mathematics is the study of integer or rational points on algebraic varieties. Let $V \subset \mathbb{P}^n$ be a projective variety that is cut out by a finite system of homogeneous equations defined over $\mathbb{Q}$. Then there are a number of basic questions that can be asked about the set $V(\mathbb{Q}) := V \cap \mathbb{P}^n(\mathbb{Q})$ of rational points on V: when is $V(\mathbb{Q})$ non-empty? how large is $V(\mathbb{Q})$ when it is non-empty? This paper aims to survey the second question, for a large class of varieties V for which one expects $V(\mathbb{Q})$ to be Zariski dense in V.

To make sense of this it is convenient to define the *height* of a projective rational point $x = [x_0, \ldots, x_n] \in \mathbb{P}^n(\mathbb{Q})$ to be $H(x) := \|\mathbf{x}\|$, for any norm $\|\cdot\|$ on $\mathbb{R}^{n+1}$, provided that $\mathbf{x} = (x_0, \ldots, x_n) \in \mathbb{Z}^{n+1}$ and $\gcd(x_0, \ldots, x_n) = 1$. Throughout this work we shall work with the height metrized by the choice of norm $|\mathbf{x}| := \max_{0 \leqslant i \leqslant n} |x_i|$. Given a suitable Zariski open subset $U \subseteq V$, the goal is then to study the quantity

$$N_{U,H}(B) := \#\{x \in U(\mathbb{Q}) : H(x) \leqslant B\}, \tag{1}$$

as $B \to \infty$. It is natural to question whether the asymptotic behaviour of $N_{U,H}(B)$ can be related to the geometry of V, for suitable open subsets $U \subseteq V$. Around 1989 Manin initiated a program to do exactly this for varieties with ample anticanonical divisor [**FMT89**]. Suppose for simplicity that $V \subset \mathbb{P}^n$ is a non-singular complete intersection, with $V = W_1 \cap \cdots \cap W_t$ for hypersurfaces $W_i \subset \mathbb{P}^n$ of degree d_i. Since V is assumed to be Fano, its Picard group is a finitely generated free $\mathbb{Z}$-module, and we denote its rank by ρ_V. In this setting the Manin conjecture takes the following shape [**BM90**, Conjecture C'].

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Conjecture A. *Suppose that $d_1+\cdots+d_t \leqslant n$. Then there exists a Zariski open subset $U \subseteq V$ and a non-negative constant $c_{V,H}$ such that*

$$N_{U,H}(B) = c_{V,H}B^{n+1-d_1-\cdots-d_t}(\log B)^{\rho_V-1}\big(1+o(1)\big), \tag{2}$$

as $B\to\infty$.

It should be noted that there exist heuristic arguments supporting the value of the exponents of B and $\log B$ appearing in the conjecture [**SD04**, §8]. The constant $c_{V,H}$ has also received a conjectural interpretation at the hands of Peyre [**Pey95**], and this has been generalised to cover certain other cases by Batyrev and Tschinkel [**BT98b**], and Salberger [**Sal98**]. In fact whenever we refer to the Manin conjecture we shall henceforth mean that the value of the constant $c_{V,H}$ should agree with the prediction of Peyre et al. With this in mind, the Manin conjecture can be extended to cover certain other Fano varieties V which are not necessarily complete intersections, nor non-singular. For the former one simply takes the exponent of B to be the infimum of numbers $a/b\in\mathbb{Q}$ such that $b>0$ and $aH+bK_V$ is linearly equivalent to an effective divisor, where $K_V\in\mathrm{Div}(V)$ is a canonical divisor and $H\in\mathrm{Div}(V)$ is a hyperplane section. For the latter, if V has only rational double points one may apply the conjecture to a minimal desingularisation $\widetilde{V}$ of V, and then use the functoriality of heights. A discussion of these more general versions of the conjecture can be found in the survey of Tschinkel [**Tsc03**]. The purpose of this note is to give an overview of our progress in the case that V is a suitable Fano variety of dimension 2.

Let $d\geqslant 3$. A non-singular surface $S\subset\mathbb{P}^d$ of degree d, with very ample anticanonical divisor $-K_S$, is known as a *del Pezzo surface of degree d*. Their geometry has been expounded by Manin [**Man86**], for example. It is well-known that such surfaces S arise either as the quadratic Veronese embedding of a quadric in $\mathbb{P}^3$, which is a del Pezzo surface of degree 8 in $\mathbb{P}^8$ (isomorphic to $\mathbb{P}^1\times\mathbb{P}^1$), or as the blow-up of $\mathbb{P}^2$ at $9-d$ points in general position, in which case the degree of S satisfies $3\leqslant d\leqslant 9$. Apart from a brief mention in the final section of this paper, we shall say nothing about del Pezzo surfaces of degree 1 or 2 in this work. The arithmetic of such surfaces remains largely elusive.

We proceed under the assumption that $3\leqslant d\leqslant 9$. In terms of the expected asymptotic formula for $N_{U,H}(B)$ for a suitable open subset $U\subseteq S$, the exponent of B is 1, and the exponent of $\log B$ is at most $9-d$, since the geometric Picard group $\mathrm{Pic}(S\otimes_{\mathbb{Q}}\overline{\mathbb{Q}})$ has rank $10-d$. An old result of Segre ensures that the set $S(\mathbb{Q})$ of rational points on S is Zariski dense as soon as it is non-empty. Moreover, S may contain certain so-called *accumulating subvarieties* that can dominate the behaviour of the counting function $N_{S,H}(B)$. These are the possible lines contained in S, whose configuration is intimately related to the configuration of points in the plane that are blown-up to obtain S. Now it is an easy exercise to check that

$$N_{\mathbb{P}^1,H}(B) = \frac{12}{\pi^2}B^2\big(1+o(1)\big),$$

as $B\to\infty$, so that $N_{V,H}(B)\gg_V B^2$ for any geometrically integral surface $V\subset\mathbb{P}^n$ that contains a line defined over $\mathbb{Q}$. However, if $U\subseteq V$ is defined to be the Zariski open subset formed by deleting all of the lines from V then it follows from combining an estimate of Heath-Brown [**HB02**, Theorem 6] with a simple birational projection argument, that $N_{U,H}(B)=o_V(B^2)$.

Returning to the setting of del Pezzo surfaces $S \subset \mathbb{P}^d$ of degree d, it turns out that there are no accumulating subvarieties when $d = 9$, or when $d = 8$ and S is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$, in which case we study $N_{S,H}(B)$. When $3 \leqslant d \leqslant 7$, or when $d = 8$ and S is not isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$, there are a finite number of accumulating subvarieties, equal to the lines in S. In these cases we study $N_{U,H}(B)$ for the open subset U formed by deleting all of the lines from S. We now proceed to review the progress that has been made towards the Manin conjecture for del Pezzo surfaces of degree $d \geqslant 3$. In doing so we have split our discussion according to the degree of the surface. It will become apparent that the problem of estimating $N_{U,H}(B)$ becomes harder as the degree decreases.

1.1. Del Pezzo surfaces of degree $\geqslant 5$. It turns out that the non-singular del Pezzo surfaces S of degree $d \geqslant 6$ are toric, in the sense that they contain the torus $\mathbb{G}_m^2$ as a dense open subset, whose natural action on itself extends to all of S. Thus the Manin conjecture for such surfaces is a special case of the more general work due to Batyrev and Tschinkel [**BT98a**], that establishes this conjecture for all toric varieties. One may compare this result with the work of de la Bretèche [**dlB01**] and Salberger [**Sal98**], who both establish the conjecture for toric varieties defined over $\mathbb{Q}$, and also the work of Peyre [**Pey95**], who handles a number of special cases.

For non-singular del Pezzo surfaces $S \subset \mathbb{P}^5$ of degree 5, the situation is rather less satisfactory. In fact there are very few instances for which the Manin conjecture has been established. The most significant of these is due to de la Bretèche [**dlB02**], who has proved the conjecture when the 10 lines are all defined over $\mathbb{Q}$. In such cases we say that the surface is *split* over $\mathbb{Q}$. Let S_0 be the surface obtained by blowing up $\mathbb{P}^2$ along the four points

$$p_1 = [1,0,0], \quad p_2 = [0,1,0], \quad p_3 = [0,0,1], \quad p_4 = [1,1,1],$$

and let $U_0 \subset S_0$ denote the corresponding open subset formed by deleting the lines from S_0. Then $\mathrm{Pic}(S_0)$ has rank 5, since S_0 is split over $\mathbb{Q}$, and de la Bretèche obtains the following result.

THEOREM 1. *Let $B \geqslant 3$. Then there exists a constant $c_0 > 0$ such that*

$$N_{U_0,H}(B) = c_0 B (\log B)^4 \Big(1 + O\Big(\frac{1}{\log\log B}\Big)\Big).$$

We shall return to the proof of this result below. The other major achievement in the setting of quintic del Pezzo surfaces is a result of de la Bretèche and Fouvry [**dlBF04**]. Here the Manin conjecture is established for the surface obtained by blowing up $\mathbb{P}^2$ along four points in general position, two of which are defined over $\mathbb{Q}$ and two of which are conjugate over $\mathbb{Q}(i)$. In related work, Browning [**Bro03b**] has obtained upper bounds for $N_{U,H}(B)$ that agree with the Manin prediction for several other del Pezzo surfaces of degree 5.

1.2. Del Pezzo surfaces of degree 4. A quartic del Pezzo surface $S \subset \mathbb{P}^4$, that is defined over $\mathbb{Q}$, can be recognised as the zero locus of a suitable pair of quadratic forms $Q_1, Q_2 \in \mathbb{Z}[x_0, \ldots, x_4]$. Then $S = \mathrm{Proj}(\mathbb{Q}[x_0, \ldots, x_4]/(Q_1, Q_2))$ is the complete intersection of the hypersurfaces $Q_1 = 0$ and $Q_2 = 0$ in $\mathbb{P}^4$. When S is non-singular (2) predicts the existence of a constant $c_{S,H} \geqslant 0$ such that

$$N_{U,H}(B) = c_{S,H} B (\log B)^{\rho_S - 1}\big(1 + o(1)\big), \tag{3}$$

as $B \to \infty$, where $\rho_S = \operatorname{rk}\operatorname{Pic}(S) \leqslant 6$ and $U \subset S$ is obtained by deleting the 16 lines from S. In this setting the best result available is due to Salberger. In work communicated at the conference *Higher dimensional varieties and rational points* at Budapest in 2001, he establishes the estimate $N_{U,H}(B) = O_{\varepsilon,S}(B^{1+\varepsilon})$ for any $\varepsilon > 0$, provided that the surface contains a conic defined over $\mathbb{Q}$. In fact an examination of Salberger's approach, which is based upon fibering the surface into a family of conics, reveals that one can replace the factor B^{ε} by $(\log B)^A$ for a large constant A. It would be more interesting to find examples of surfaces S for which the exponent A could be reduced to the expected quantity $\rho_S - 1$.

It emerges that much more can be said if one permits S to contain isolated singularities. For the remainder of this section let $S \subset \mathbb{P}^4$ be a geometrically integral intersection of two quadric hypersurfaces, which has only isolated singularities and is not a cone. Then S contains only rational double points (see Wall [**Wal80**], for example), thereby ensuring that there exists a unique minimal desingularisation $\pi : \widetilde{S} \to S$ of the surface, such that $K_{\widetilde{S}} = \pi^* K_S$. In particular it follows that the asymptotic formula (3) is still expected to hold, with ρ_S now taken to be the rank of the Picard group of $\widetilde{S}$, and $U \subset S$ obtained by deleting all of the lines from S. The classification of such surfaces S is rather classical, and can be found in the work of Hodge and Pedoe [**HP52**, Book IV, §XIII.11], for example. It turns out that up to isomorphism over $\overline{\mathbb{Q}}$, there are 15 possible singularity types for S, each categorised by the *extended Dynkin diagram*. This is the Dynkin diagram that describes the intersection behaviour of the exceptional divisors and the transforms of the lines on the minimal desingularisation $\widetilde{S}$ of S. Of course, if one is interested in a classification over the ground field $\mathbb{Q}$, then many more singularity types can occur (see Lipman [**Lip69**], for example). Over $\overline{\mathbb{Q}}$, Coray and Tsfasman [**CT88**, Proposition 6.1] have calculated the extended Dynkin diagrams for all of the 15 types, and this information allows us to write down a list of surfaces $S = \operatorname{Proj}(\mathbb{Q}[\mathbf{x}]/(Q_1, Q_2))$ that typify each possibility, together with their singularity type and the number of lines that they contain. The author is grateful to Ulrich Derenthal for helping to prepare the following table.

type	$Q_1(\mathbf{x})$	$Q_2(\mathbf{x})$	# lines	singularity
i	$x_0x_1 + x_2x_3$	$x_0x_3 + x_1x_2 + x_2x_4 + x_3x_4$	12	$\mathbf{A}_1$
ii	$x_0x_1 + x_2x_3$	$x_0x_3 + x_1x_2 + x_2x_4 + x_4^2$	9	$2\mathbf{A}_1$
iii	$x_0x_1 + x_2^2$	$x_0x_2 + x_1x_2 + x_3x_4$	8	$2\mathbf{A}_1$
iv	$x_0x_1 + x_2x_3$	$x_2x_3 + x_4(x_0 + x_1 + x_2 - x_3)$	8	$\mathbf{A}_2$
v	$x_0x_1 + x_2^2$	$x_1x_2 + x_2^2 + x_3x_4$	6	$3\mathbf{A}_1$
vi	$x_0x_1 + x_2x_3$	$x_1^2 + x_2^2 + x_3x_4$	6	$\mathbf{A}_1 + \mathbf{A}_2$
vii	$x_0x_1 + x_2x_3$	$x_1x_3 + x_2^2 + x_4^2$	5	$\mathbf{A}_3$
viii	$x_0x_1 + x_2^2$	$(x_0 + x_1)^2 + x_2x_4 + x_3^2$	4	$\mathbf{A}_3$
ix	$x_0x_1 + x_2^2$	$x_2^2 + x_3x_4$	4	$4\mathbf{A}_1$
x	$x_0x_1 + x_2^2$	$x_1x_2 + x_3x_4$	4	$2\mathbf{A}_1 + \mathbf{A}_2$
xi	$x_0x_1 + x_2^2$	$x_0^2 + x_2x_4 + x_3^2$	3	$\mathbf{A}_1 + \mathbf{A}_3$
xii	$x_0x_1 + x_2x_3$	$x_0x_4 + x_1x_3 + x_2^2$	3	$\mathbf{A}_4$
xiii	$x_0x_1 + x_2^2$	$x_0^2 + x_1x_4 + x_3^2$	2	$\mathbf{D}_4$
xiv	$x_0x_1 + x_2^2$	$x_0^2 + x_3x_4$	2	$2\mathbf{A}_1 + \mathbf{A}_3$
xv	$x_0x_1 + x_2^2$	$x_0x_4 + x_1x_2 + x_3^2$	1	$\mathbf{D}_5$