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# An Introduction to Heegaard Floer Homology

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## 1. Introduction

The aim of these notes is to give an introduction to Heegaard Floer homology for closed oriented 3-manifolds [**31**]. We will also discuss a related Floer homology invariant for knots in $S^3$ [**29**], [**34**].

Let $Y$ be an oriented closed 3-manifold. The simplest version of Heegaard Floer homology associates to $Y$ a finitely generated Abelian group $\widehat{HF}(Y)$. This homology is defined with the help of Heegaard diagrams and Lagrangian Floer homology. Variants of this construction give related invariants $HF^+(Y)$, $HF^-(Y)$, $HF^\infty(Y)$.

While its construction is very different, Heegaard Floer homology is closely related to Seiberg-Witten Floer homology [**10, 15, 17**], and instanton Floer homology [**3, 4, 7**]. In particular it grew out of our attempt to find a more topological description of Seiberg-Witten theory for three-manifolds.

## 2. Heegaard decompositions and diagrams

Let $Y$ be a closed oriented three-manifold. In this section we describe decompositions of $Y$ into more elementary pieces, called handlebodies.

A *genus g handlebody* $U$ is diffeomorphic to a regular neighborhood of a bouquet of $g$ circles in $\mathbb{R}^3$; see Figure 1. The boundary of $U$ is an oriented surface of genus $g$. If we glue two such handlebodies together along their common boundary, we get a closed 3-manifold

$$Y = U_0 \cup_\Sigma U_1$$

oriented so that $\Sigma$ is the oriented boundary of $U_0$. This is called a Heegaard decomposition for $Y$.

**2.1. Examples.** The simplest example is the (genus 0) decomposition of $S^3$ into two balls. A similar example is given by taking a tubular neighborhood of the unknot in $S^3$. Since the complement is also a solid torus, we get a genus 1 Heegaard decomposition of $S^3$.

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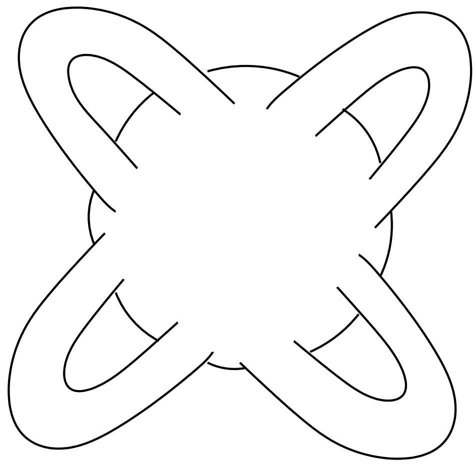

FIGURE 1. A handlebody of genus 4.

Other simple examples are given by lens spaces. Take

$$S^3 = \{(z,w) \in \mathbb{C}^2 |\ |z^2| + |w|^2 = 1\}$$

Let $(p,q)=1$, $1 \leq q < p$. The lens space $L(p.q)$ is given by dividing out $S^3$ by the free $\mathbb{Z}/p$ action

$$f : (z,w) \longrightarrow (\alpha z, \alpha^q w),$$

where $\alpha = e^{2\pi i/p}$. Clearly $\pi_1(L(p,q)) = \mathbb{Z}/p$. Note also that the solid tori $U_0 = \{|z| \leq \frac{1}{2}\ \}$, $U_1 = \{|z| \geq \frac{1}{2}\}$ are preserved by the action, and their quotients are also solid tori. This gives a genus 1 Heegaard decomposition of $L(p,q)$.

**2.2. Existence of Heegaard decompositions.** While the small genus examples might suggest that 3-manifolds that admit Heegaard decompositions are special, in fact the opposite is true:

THEOREM 2.1. *([**39**]) Let $Y$ be an oriented closed three-dimensional manifold. Then $Y$ admits a Heegaard decomposition.*

PROOF. Start with a triangulation of $Y$. The union of the vertices and the edges gives a graph in $Y$. Let $U_0$ be a small neighborhood of this graph. In other words replace each vertex by a ball, and each edge by a solid cylinder. By definition $U_0$ is a handlebody. It is easy to see that $Y - U_0$ is also a handlebody, given by a regular neighborhood of a graph on the centers of the triangles and tetrahedra in the triangulation. □

**2.3. Stabilizations.** It follows from the above proof that the same three-manifold admits lots of different Heegaard decompositions. In particular, given a Heegaard decomposition $Y = U_0 \cup_\Sigma U_1$ of genus $g$, we can define another decomposition of genus $g+1$ by choosing two points in $\Sigma$ and connecting them by a small unknotted arc $\gamma$ in $U_1$. Let $U_0'$ be the union of $U_0$ and a small tubular neighborhood $N$ of $\gamma$. Similarly let $U_1' = U_1 - N$. The new decomposition

$$Y = U_0' \cup_{\Sigma'} U_1'$$

is called the *stabilization* of $Y = U_0 \cup_\Sigma U_1$. Clearly $g(\Sigma') = g(\Sigma) + 1$. For an easy example note that the genus 1 decomposition of $S^3$ described earlier is the stabilization of the genus 0 decomposition.

According to a theorem of Singer [**39**], any two Heegaard decompositions can be connected by stabilizations (and destabilizations):

THEOREM 2.2. *Let $(Y, U_0, U_1)$ and $(Y, U_0', U_1')$ be two Heegaard decompositions of $Y$ of genus $g$ and $g'$ respectively. Then for $k$ large enough the $(k - g')$-fold stabilization of the first decomposition is diffeomorphic to the $(k - g)$-fold stabilization of the second decomposition.*

**2.4. Heegaard diagrams.** In view of Theorem 2.2, if we find an invariant for Heegaard decompositions with the property that it does not change under stabilization, then this is in fact a three-manifold invariant. For example the Casson invariant [**1, 37**] is defined in this way. However, for the definition of Heegaard Floer homology we need some additional information which is given by diagrams.

Let us start with a handlebody $U$ of genus $g$.

DEFINITION 2.3. A set of attaching circles $(\gamma_1, ..., \gamma_g)$ for $U$ is a collection of closed embedded curves in $\Sigma_g = \partial U$ with the following properties

- The curves $\gamma_i$ are disjoint from each other.
- $\Sigma_g - \gamma_1 - \cdots - \gamma_g$ is connected.
- The curves $\gamma_i$ bound disjoint embedded disks in $U$.

REMARK 2.4. The second property in the above definition is equivalent to the property that $([\gamma_1], ..., [\gamma_g])$ are linearly independent in $H_1(\Sigma, \mathbb{Z})$.

DEFINITION 2.5. Let $(\Sigma_g, U_0, U_1)$ be a genus $g$ Heegaard decomposition for $Y$. A *compatible Heegaard diagram* is given by $\Sigma_g$ together with a collection of curves $\alpha_1, ..., \alpha_g, \beta_1, ..., \beta_g$ with the property that $(\alpha_1, ..., \alpha_g)$ is a set of attaching circles for $U_0$ and $(\beta_1, ..., \beta_g)$ is a set of attaching circles for $U_1$.

REMARK 2.6. A Heegaard decomposition of genus $g > 1$ admits lots of different compatible Heegaard diagrams.

In the opposite direction any diagram $(\Sigma_g, \alpha_1, ..., \alpha_g, \beta_1, ..., \beta_g)$ where the $\alpha$ and $\beta$ curves satisfy the first two conditions in Definition 2.3 determines uniquely a Heegaard decomposition and therefore a 3-manifold.

**2.5. Examples.** It is helpful to look at a few examples. The genus 1 Heegaard decomposition of $S^3$ corresponds to a diagram $(\Sigma_1, \alpha, \beta)$ where $\alpha$ and $\beta$ meet transversely in a unique point. $S^1 \times S^2$ corresponds to $(\Sigma_1, \alpha, \alpha)$.

The lens space $L(p, q)$ has a diagram $(\Sigma_1, \alpha, \beta)$ where $\alpha$ and $\beta$ intersect at $p$ points and in a standard basis $x, y \in H_1(\Sigma_1) = \mathbb{Z} \oplus \mathbb{Z}$, $[\alpha] = y$ and $[\beta] = px + qy$.

Another example is given in Figure 2. Here we think of $S^2$ as the plane together with the point at infinity. In the picture the two circles on the left are identified, or equivalently we glue a handle to $S^2$ along these circles. Similarly we identify the two circles on the right side of the picture. After this identification the two horizontal lines become closed circles $\alpha_1$ and $\alpha_2$. As for the two $\beta$ curves, $\beta_1$ lies in the plane and $\beta_2$ goes through both handles once.

DEFINITION 2.7. We can define a one-parameter family of Heegaard diagrams by changing the right side of Figure 2. For $n > 0$ instead of twisting around the