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Contact Surgery and Heegaard Floer Theory

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Abstract. The fundamental theorem of Giroux — relating contact structures and open book decompositions — provides a way to study contact structures on closed 3–manifolds from a topological point of view. Contact surgery diagrams allow us to use some form of Kirby calculus in the study of contact 3–manifolds, and Heegaard Floer theory — through the Ozsváth–Szabó knot invariant of the binding of a compatible open book decomposition — gives a very sensitive contact invariant, which seems to be crucial in attacking the classification problem of tight contact structures on certain types of closed 3–manifolds. In these notes we collected the basic ideas of contact surgery and computation of contact Ozsváth–Szabó invariants. We paid special attention to explicit computations, hoping to convince the reader that the usual Heegaard Floer package, together with some simple homotopy–theoretic arguments might be used to derive exciting new results in contact topology.

1. Contact 3–manifolds

General definitions. We start our discussion by recalling basic notions of contact topology — for a more complete treatment of the topics just mentioned here, see [**7, 11**].

Let Y be a given closed, oriented, smooth 3–manifold. A 1–form α is a (positive) *contact form* if $\alpha \wedge d\alpha > 0$ (with respect to the given orientation). A 2–plane field ξ is a positive, coorientable contact structure if there is a contact 1–form $\alpha \in \Omega^1(Y)$ such that $\xi = \ker \alpha$. By fixing α (up to multiplication by smooth functions $f\colon Y \to \mathbb{R}^+$) we also fix an orientation for the 2–plane field ξ: the basis $\{v_1, v_2\} \subset \xi_p$ is positive if $\{v_1, v_2, n\}$ with normal vector n satisfying $\alpha(n) > 0$ provides an oriented basis for T_pY. In this case the contact structure is *cooriented*.

Let (X, ω) be a given compact, symplectic 4–manifold, that is, X is a smooth, compact, oriented 4–manifold with possibly non–empty boundary and ω is a closed 2–form with $\omega \wedge \omega > 0$ (with respect to the given orientation). The contact 3–manifold (Y, ξ) is *compatible* with (X, ω), or (X, ω) is a *filling* of (Y, ξ) if $\partial X = Y$ as oriented manifolds and $\omega|_\xi \neq 0$. In this case (X, ω) is also called a *weak symplectic filling* of (Y, ξ).

A symplectic filling (X, ω) is a *strong filling* of (Y, ξ) if ω is exact near $\partial X = Y$ and there is a 1-form α near ∂X with $\omega = d\alpha$, $d\alpha|_\xi \neq 0$ and $\xi = \{\alpha|_Y = 0\}$. It can

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be shown that the existence of such α is equivalent to the existence of a vector field v defined near ∂X which is transverse to the boundary and is a symplectic dilation, that is, $\mathfrak{L}_v\omega = \omega$. In particular, for a strong filling (X, ω) the symplectic structure on a collar of the boundary can be shown to have a model in a symplectic manifold (in the *symplectization* of (Y, ξ)) which depends only on the contact structure ξ; therefore strong fillings are suitable for performing symplectic surgeries. Notice that if ω is nonexact near ∂X then (X, ω) is not a strong filling. It turns out that this is the only obstruction, more precisely

LEMMA 1.1 (Eliashberg [**6**], Ohta–Ono [**24**]). *If (X, ω) is a weak filling of (Y, ξ) and ω is exact on a collar neighbourhood of ∂X then ω can be perturbed near ∂X to a symplectic form $\tilde\omega$ such that $(X, \tilde\omega)$ is a strong filling of (Y, ξ).* □

Since on a rational homology 3–sphere any 2–form is exact, this implies

COROLLARY 1.2. *Suppose that Y is a rational homology 3–sphere, i.e., $b_1(Y) = 0$. If (X, ω) is a weak filling of (Y, ξ) for some contact structure ξ then ω can be perturbed to provide a strong filling $(X, \tilde\omega)$ of (Y, ξ).* □

The compact complex manifold (X, J) with complex structure J is a *Stein filling* of (Y, ξ) if $\partial X = Y$, ξ is given as the oriented 2–plane field of complex tangencies on Y and (X, J) is a Stein domain, that is, it admits a proper function $\varphi\colon X \to [0, \infty)$ with $\partial X = \varphi^{-1}(a)$ for some regular value $a \in \mathbb{R}$ which is *plurisubharmonic*, i.e., the 2–form $\omega_\varphi = -d^{\mathbb{C}}d\varphi$ is a Kähler form with associated Kähler metric g_φ. It is not hard to see that a Stein filling is always a strong filling and a strong filling is automatically a weak filling. For more about fillings see [**6**].

EXAMPLE 1.3. It is easy to see that the 1–form $\alpha = dz + xdy$ induces a contact structure on the 3–dimensional Euclidean space $\mathbb{R}^3$. It turns out that this contact structure extends to the 3–sphere S^3. In addition, the resulting 2–plane field is isotopic to the 2–plane field of complex tangencies on S^3 when viewed as the boundary of the unit 4–ball in the complex vector space $\mathbb{C}^2$. The above structures are the *standard* contact structures on $\mathbb{R}^3$ and S^3, and we will denote them by ξ_{st}.

A knot $K \subset (Y, \xi)$ is called *Legendrian* if it is tangent to ξ, i.e., if ξ is defined by the 1–form α then $\alpha(TK) = 0$. Every knot can be smoothly isotoped to a Legendrian knot, in fact, for every knot there is a C^0–close Legendrian knot smoothly isotopic to it.

Legendrian knots in $(\mathbb{R}^3, \xi_{st})$ (and so in (S^3, ξ_{st})) can be depicted by their *front projections* to the yz–plane, since according to the equation $x = -\frac{dz}{dy}$ the slope of the tangent of the front projection determines the x–coordinate. After possibly isotoping, every Legendrian knot admits a front projection with no triple points, transverse double points and $(2, 3)$-cusps instead of vertical tangencies. Conversely, any front projection having cusps instead of vertical tangencies and not admitting crossings with higher slope in front uniquely specifies a Legendrian knot. For this reason we will symbolize Legendrian knots in $(\mathbb{R}^3, \xi_{st})$ (and so in (S^3, ξ_{st})) by their front projections.

Notice that if $L \subset (Y, \xi)$ is Legendrian, it admits a canonical framing: consider the unit orthogonal of the tangent vector of L in ξ. (When choosing the particular orthogonal, we take the orientation of the 2–plane field into account.) The resulting framing is called the *contact framing* of the Legendrian knot L. If L is null-homologous in Y then it admits another framing, induced by pushing off L

along its existing Seifert surface. This latter framing is called the *Seifert framing.* When measuring the contact framing with respect to this Seifert framing we get an integer invariant of the Legendrian knot L called the *Thurston–Bennequin invariant* $\mathrm{tb}(L)$. Notice that since the Seifert framing is well–defined and independent of the chosen Seifert surface, the Thurston–Bennequin invariant depends only on the Legendrian knot L. Since knots in $\mathbb{R}^3$ and S^3 are all null–homologous, they all admit Thurston–Bennequin invariants. The computation of $\mathrm{tb}(L)$ from a front projection of L is an easy task: it is equal to

$$w(L) - \tfrac{1}{2}c(L),$$

where $w(L)$ is the *writhe* of the projection, i.e. the signed number of the double points of the projection, and $c(L)$ is the number of cusps in the projection. Since left and right cusps alternate among each other, it is easy to see that $\frac{1}{2}c(L) = c_r(L) = c_l(L)$ where $c_r(L)$ (and $c_l(L)$) stands for the right (resp. left) cusps of the projection.

Example 1.4. Figure 1 shows the front projection of a Legendrian knot smoothly isotopic to the right–handed trefoil. The writhe of this projection is 3, and has 4 cusps, hence the Thurston–Bennequin invariant of the Legendrian knot determined by the front projection is equal to 1.

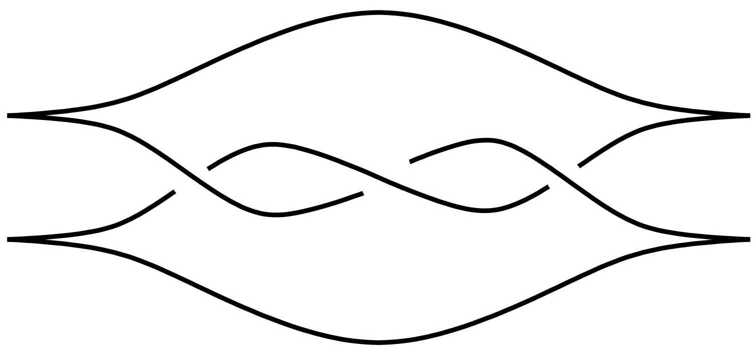

Figure 1. Front projection of a Legendrian trefoil knot

If $L \subset (Y, \xi)$ is null-homologous then there is another numerical invariant we can associate to it: consider a Seifert surface $\Sigma \subset (Y, \xi)$ and take the relative Euler class of ξ (as a 2–plane bundle) over Σ. For this to make sense we need to trivialize ξ over $\partial\Sigma = L$: choose the trivialization provided by the tangents of L together with their oriented normals in ξ. Note that in order to specify the tangents we need to fix an orientation on L. It is not hard to see that the resulting quantity, called the *rotation number* $\mathrm{rot}_\Sigma(L)$, will depend on the chosen Seifert surface and the orientation fixed on the knot. If $L \subset \mathbb{R}^3$ or S^3, however, the vanishing of the second homology group implies that the rotation number is independent of the chosen Seifert surface. If L is in $\mathbb{R}^3$ or in S^3, the rotation number can be computed for L given by a front projection by the formula

$$\mathrm{rot}(L) = \tfrac{1}{2}(c_d(L) - c_u(L)),$$

where $c_u(L)$ (resp. $c_d(L)$) denotes the number of up (resp. down) cusps of the projection.