Clay Mathematics Proceedings
Volume 5, 2006

Monodromy, Vanishing Cycles, Knots and the Adjoint Quotient

Ivan Smith

Abstract. After reviewing some (mostly standard) material on symplectic fibre bundles, we describe a cohomology theory for oriented links in the three-sphere. This cohomological invariant, introduced in joint work with Paul Seidel, is defined by combining results from Lie algebra theory with Lagrangian Floer cohomology, and conjecturally equals Khovanov cohomology after collapsing the latter's grading.

These notes, are divided into two parts. The first part describes as background some of the geometry of symplectic fibre bundles and their monodromy. The second part applies these general ideas to certain Stein fibre bundles that arise naturally in Lie theory, to construct an invariant of oriented links in the three-sphere (Section 2.8). Despite its very different origins, this invariant is conjecturally equal to the combinatorial homology theory defined by Mikhail Khovanov (Section 2.9). In the hope of emphasising the key ideas, concision has taken preference over precision; there are no proofs, and sharp(er) forms of statements are deferred to the literature.

Much of the first part I learned from, and the second part represents joint work with, Paul Seidel, whose influence and insights generously pervade all that follows.

1. Monodromy, vanishing cycles

Most of the material in this section is well-known; general references are [**16**], [**23**],[**5**].

1.1. Symplectic fibre bundles: We will be concerned with fibrations $p : X \to B$ with symplectic base and fibre, or more precisely where X carries a closed vertically non-degenerate 2-form Ω, for which $d\Omega(u, v, \cdot) = 0$ whenever u, v are vertical tangent vectors. If the fibration is proper, the cohomology class $[\Omega|_{\mathrm{Fibre}}]$ is locally constant, and parallel transport maps are symplectomorphisms. Examples abound:

(1) A surface bundle over any space $\Sigma_g \to X \to B$ with fibre essential in homology can be given this structure; define Ω by picking any 2-form dual to the fibre and whose restriction to each fibre is an area form. The homology constraint is

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1991 *Mathematics Subject Classification.* 53D35, 57M25.

Thanks to Aaron Lauda for help with the pictures. This work was partially supported by a grant NUF-NAL/00876/G from the Nuffield Foundation.

automatically satisfied whenever $g \geq 2$ (evaluate the first Chern class of the vertical tangent bundle on a fibre).

(2) Given a holomorphic map $p : X \to B$ defined on a quasiprojective variety and which is smooth over $B^0 \subset B$, the restriction $p^{-1}(B^0) \to B^0$ defines a symplectic fibre bundle, where the 2-form Ω is the restriction of a Kähler form on X. Such examples show the importance of *singular fibres*. Rational maps and linear systems in algebraic geometry provide a plethora of interesting singular fibrations.

(3) Contrastingly, the (singular) fibrations arising from moment maps, cotangent bundle projections and many dynamical systems have Lagrangian fibres, and fall outside the scope of the machinery we'll discuss.

Strictly, it is sensible to make a distinction between Hamiltonian and more general symplectic fibrations; essentially this amounts to the question of whether the vertically non-degenerate 2-form Ω has a closed extension to the total space, as in the cases above. The subtlety will not play any role in what follows, but for discussion see [**16**].

1.2. Parallel transport: A symplectic fibre bundle has a distinguished connexion, where the horizontal subspace at $x \in X$ is the symplectic orthogonal complement to the vertical distribution $\mathrm{Hor}_x = \ker(dp_x)^{\perp\Omega}$. For Kähler fibrations, since the fibres are complex submanifolds, we can also define this as the orthogonal complement to $\ker(dp_x)$ with respect to the Kähler metric. We should emphasise at once that, in contrast to the Darboux theorem which prevents local curvature-type invariants entering symplectic topology naively, there is no "universal triviality" result for symplectic fibre bundles. The canonical connexion can, and often does, have curvature, and that curvature plays an essential part in the derivations of some of the theorems of the sequel.

Given a path $\gamma : [0,1] \to B$ we can lift the tangent vector $d\gamma/dt$ to a horizontal vector field on $p^{-1}(\gamma)$, and flowing along the integral curves of this vector field defines local *symplectomorphisms* h^γ of the fibres.

(1) If $p : X \to B$ is proper, the horizontal lifts can be globally integrated and we see that p is a fibre bundle with structure group $\mathrm{Symp}(p^{-1}(b))$. Note that, since the connexion isn't flat, the structure group does not in general reduce to the symplectic mapping class group (of components of Symp).

(2) Often there is a group G acting fibrewise and preserving all the structure, in which case parallel transport will be G-equivariant. An example will be given shortly.

1.3. Non-compactness: If the fibres are not compact, the local parallel transport maps may not be globally defined, since the solutions to the differential equations defining the integral curves may not exist for all times. To overcome this, there are several possible strategies. The simplest involves estimating the parallel transport vector fields explicitly (which in turn might rely on choosing the right Ω).

Suppose for instance $p : X \to \mathbb{C}$ where X has a Kähler metric; then for $V \in T_{p(x)}(\mathbb{C})$ the lift $V^{hor} = V.\frac{(\nabla p)_x}{|\nabla p|^2}$. If $p : (\mathbb{C}^n, \omega_{st}) \to \mathbb{C}$ is a homogeneous polynomial, clearly its only critical value is the origin, giving a fibre bundle over $\mathbb{C}^*$. The identity $dp_x(x) = deg(p).p(x)$, together with the previous formula for V^{hor}, shows that the horizontal lift of a tangent vector $V \in T_p(x)(\mathbb{C}) = \mathbb{C}$ has norm $|V^{hor}| \leq \frac{|V|.|x|}{deg(p).|p(x)|}$. On a fixed fibre $p = const$ this grows linearly with $|x|$ and can be globally integrated.

Corollary: For homogeneous polynomials $p : (\mathbb{C}^n, \omega_{st}) \to \mathbb{C}$ parallel transport is globally defined over $\mathbb{C}^*$.

Example: the above applies to the determinant mapping (indeed any single component of the characteristic polynomial or *adjoint quotient* map), $\det : \mathrm{Mat}_n(\mathbb{C}) \to \mathbb{C}$. In this case, parallel transport is invariant under $SU_n \times SU_n$. The monodromy of the associated bundle seems never to have been investigated.

For mappings $p : \mathbb{C}^n \to \mathbb{C}^m$ in which each component is a homogeneous polynomial but the homogeneous degrees differ, the above arguments do not quite apply but another approach can be useful. Since the smooth fibres are Stein manifolds of finite type, we can find a vector field Z which points inwards on all the infinite cones. By flowing with respect to a vector field $V^{hor} - \delta Z$, for large enough δ, and then using the Liouville flows, we can define "rescaled" parallel transport maps $h^{\gamma}_{resc} : p^{-1}(t) \cap B(R) \hookrightarrow p^{-1}(t')$ on arbitrarily large pieces of a fixed fibre $p^{-1}(t)$, which embed such compacta symplectically into another fibre $p^{-1}(t')$. This is not quite the same as saying that the fibres are globally symplectomorphic, but is enough to transport closed Lagrangian submanifolds around (uniquely up to isotopy), and often suffices in applications. For a detailed discussion, see [**29**]. (In fact, if the Stein fibres are finite type and complete one can "uncompress" the flows above to show the fibres really are globally symplectomorphic, cf. [**13**].)

1.4. Vanishing cycles: The local geometry near a singularity (critical fibre of p) shows up in the *monodromy* of the smooth fibre bundle over B^0, i.e. the representation

$$\pi_1(B^0, b) \to \pi_0(\mathrm{Symp}(p^{-1}(b), \Omega)).$$

Consider the ordinary double point (Morse singularity, node...)

$$p : (z_1, \ldots, z_n) \mapsto \sum z_i^2.$$

The smooth fibres $p^{-1}(t)$, when equipped with the restriction of the flat Kähler form $(i/2)\sum_j dz_j \wedge d\overline{z}_j$ from $\mathbb{C}^n$, are symplectically isomorphic to $(T^*\mathbb{S}^{n-1}, \omega_{can})$. Indeed, an explicit symplectomorphism can be given in co-ordinates by viewing

$$T^*\mathbb{S}^{n-1} = \{(a, b) \in \mathbb{R}^n \times \mathbb{R}^n \mid |a| = 1, \langle a, b\rangle = 0\}$$

and taking $p^{-1}(1) \ni z \mapsto (\Re(z)/|\Re(z)|, -|\Re(z)|\Im(z))$. There is a distinguished Lagrangian submanifold of the fibre, the zero-section, which can also be defined as the locus of points which flow into the singularity under parallel transport along a radial line in $\mathbb{C}$. Accordingly, this locus – which is $\{(z_1, \ldots, z_n) \in \mathbb{R}^n \mid \sum_i |z_i|^2 = 1\}$ in co-ordinates – is also called the *vanishing cycle* of the singularity.

Lemma: The monodromy about a loop encircling $0 \in \mathbb{C}$ is a Dehn twist in the vanishing cycle.

To define the Dehn twist, fix the usual metric on $\mathbb{S}^{n-1} \subset \mathbb{R}^n$, which identifies $T^*\mathbb{S}^{n-1} \cong T\mathbb{S}^{n-1}$. The Dehn twist is the composite of the time π map of the geodesic flow on the unit disc tangent bundle $U(T\mathbb{S}^{n-1})$ with the map induced by the antipodal map; it's antipodal on the zero-section and vanishes on the boundary ∂U. If $n = 2$ this construction is classical, and we get the usual Dehn twist on a curve in an annulus $T^*\mathbb{S}^1$.