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# A Note on Symplectic 4-manifolds with $b_2^+ = 1$ and $K^2 \geq 0$

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Abstract. In this article we survey recent results on the existence problem of simply connected symplectic 4-manifolds with $b_2^+ = 1$ and $K^2 \geq 0$. We also investigate exotic smooth structures on rational surfaces $\mathbf{C}P^2 \sharp n \overline{\mathbf{C}P}^2$.

## 1. Introduction

Since S. Donaldson introduced gauge theory in the study of smooth 4-manifolds ([**DK**]), various techniques have been developed to produce new families of symplectic 4-manifolds which were not known before. For example, R. Gompf constructed many symplectic 4-manifolds using fiber-sum surgery ([**G**]), and R. Fintushel and R. Stern also constructed a family of symplectic 4-manifolds using rational blow-down surgery and 0-framed surgery ([**FS1**], [**FS2**]). Recently, using these techniques, the author constructed new simply connected symplectic 4-manifolds with $b_2^+ = 1$ and $K^2 \geq 0$ ([**P1**], [**P2**]).

The aim of this article is to survey these constructions which appeared in [**P1**] and [**P2**]. Let us start with classifying symplectic 4-manifolds with $b_2^+ = 1$. It is the usual convention that the set of symplectic 4-manifolds with $b_2^+ = 1$ is classified by the sign of the square $K^2$ of the canonical class $K$ associated to a compatible almost complex structure on a given symplectic 4-manifold. In contrast to the fact that every minimal symplectic 4-manifold with $b_2^+ > 1$ satisfies $K^2 \geq 0$ ([**T**]), there are many symplectic 4-manifolds with $b_2^+ = 1$ satisfying $K^2 < 0$, $K^2 = 0$ and $K^2 > 0$ respectively. It was known that only irrational ruled surfaces are minimal symplectic 4-manifolds with $K^2 < 0$ ([**MS**]). Next, in the category of $K^2 = 0$, most known symplectic 4-manifolds are complex surfaces such as rational or ruled surfaces, Dolgachev surfaces and Enriques surface. Even though there are some non-simply connected and non-complex symplectic 4-manifolds such as some torus bundles over the torus and $S^1 \times M$ with a fibered 3-manifold $M (\neq S^1 \times S^2)$, little has been known about simply connected minimal symplectic 4-manifolds which do not admit a complex structure. In Section 3 we confirm that most homotopy elliptic surfaces $\{E(1)_K \,|\, K \text{ is a fibered knot in } S^3\}$ constructed by R. Fintushel and R. Stern in [**FS2**] are simply connected minimal symplectic 4-manifolds which cannot admit a complex structure. The main technique involved

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in the proof is a computation of the Seiberg-Witten invariant obtained by a small generic perturbation of the Seiberg-Witten equations. Finally, in the case when $K^2 > 0$, until now the only known simply connected symplectic 4-manifolds with $b_2^+ = 1$ were rational surfaces such as $\mathbf{CP}^2, S^2 \times S^2$ and $\mathbf{C}P^2 \sharp n \overline{\mathbf{C}P}^2 (n \leq 8)$ and the Barlow surface. In Section 4 we present new simply connected symplectic 4-manifolds with $b_2^+ = 1$ and $1 \leq K^2 \leq 2$. The main technique involved in the construction is a rational blow-down surgery.

## 2. Preliminaries

In this section we briefly review Seiberg-Witten theory for smooth 4-manifolds. In particular, we focus on the Seiberg-Witten invariants of 4-manifolds with $b_2^+ = 1$ (see [**M**] for details).

Let $X$ be a closed, oriented smooth 4-manifold with $b_2^+ > 0$ and a fixed metric $g$, and let $L$ be a characteristic line bundle on $X$, i.e. $c_1(L)$ is an integral lift of $w_2(X)$ (We assume that $H_1(X;\mathbf{Z})$ has no 2-torsion.) Then $L$ determines a $Spin^c$-structure on $X$ which induces a complex spinor bundle $W \cong W^+ \oplus W^-$, where $W^{\pm}$ are the associated $U(2)$-bundles on $X$ such that $\det(W^{\pm}) \cong L$. Note that the Levi-Civita connection on $TX$ together with a unitary connection $A$ on $L$ induces a connection $\nabla_A : \Gamma(W^+) \to \Gamma(T^*X \otimes W^+)$. This connection, followed by Clifford multiplication, induces a $Spin^c$-Dirac operator $D_A : \Gamma(W^+) \to \Gamma(W^-)$. Then, for each self-dual 2-form $h \in \Omega^2_{+_g}(X : \mathbf{R})$ the following pair of equations for a unitary connection $A$ on $L$ and a section $\Psi$ of $\Gamma(W^+)$ are called the *perturbed Seiberg-Witten equations*:

$$(2.1) \qquad (SW_{g,h}) \begin{cases} D_A \Psi & = \quad 0 \\ F_A^{+_g} & = \quad i(\Psi \otimes \Psi^*)_0 + ih\,. \end{cases}$$

Here $F_A^{+_g}$ is the self-dual part of the curvature of $A$ with respect to a metric $g$ on $X$ and $(\Psi \otimes \Psi^*)_0$ is the trace-free part of $(\Psi \otimes \Psi^*)$. The gauge group $\mathcal{G} := Aut(L) \cong Map(X, S^1)$ acts on the space $\mathcal{A}_X(L) \times \Gamma(W^+)$ by

$$g \cdot (A, \Psi) = (g \circ A \circ g^{-1}, g \cdot \Psi).$$

Since the set of solutions is invariant under the action, it determines an orbit space, called the *Seiberg-Witten moduli space*, denoted by $M_{X,g,h}(L)$, whose formal dimension is

$$\dim M_{X,g,h}(L) = \frac{1}{4}(c_1(L)^2 - 3\sigma(X) - 2e(X))$$

where $\sigma(X)$ is the signature of $X$ and $e(X)$ is the Euler characteristic of $X$. Note that if $b_2^+(X) > 0$ and $M_{X,g,h}(L)$ is not empty then for a generic self-dual 2-form $h$ on $X$ the moduli space $M_{X,g,h}(L)$ contains no reducible solutions, hence it is a compact smooth manifold of the given dimension.

Definition 2.1. The *Seiberg-Witten invariant* (*in brief, SW-invariant*) of a smooth 4-manifold $X$ with $b_2^+ > 0$ is a function $SW_X : Spin^c(X) \to \mathbf{Z}$ defined by

$$(2.2) \qquad SW_X(L) := \begin{cases} <\beta^{d_L}, [M_{X,g,h}]> & \text{if } \dim M_{X,g,h}(L) := 2d_L \geq 0 \\ & \text{is nonnegative and even} \\ 0 & \text{otherwise}\,. \end{cases}$$

Here $\beta$ is a generator of $H^2(\mathcal{B}_X^*(L); \mathbf{Z})$ which is the first Chern class of the $S^1$-bundle

$$\widetilde{\mathcal{B}}_X^*(L) = \mathcal{A}_X(L) \times (\Gamma(W^+) - \{0\})/Aut^0(L) \longrightarrow \mathcal{B}_X^*(L)$$

where $Aut^0(L)$ consists of gauge transformations which are the identity on the fiber of $L$ over a fixed base point in $X$.

If $b_2^+(X) > 1$, the SW-invariant, denoted by $SW_X = \sum SW_X(L) \cdot e^{c_1(L)}$, is a diffeomorphism invariant, i.e. $SW_X$ does not depend on the choice of a metric on $X$ or a generic perturbation. Furthermore, only finitely many $Spin^c$-structures on $X$ have non-zero Seiberg-Witten invariant. We say that the characteristic line bundle $L$ or equivalently, its Chern class $c_1(L) \in H^2(X; \mathbf{Z})$, is a *SW-basic class* of $X$ if $SW_X(L) \neq 0$.

When $b_2^+(X) = 1$, the SW-invariant $SW_X(L)$ defined in (2.2) above depends not only on a metric $g$ but also on a self-dual 2-form $h$. Because of this fact, there are several types of Seiberg-Witten invariants for a smooth 4-manifold with $b_2^+ = 1$ depending on how the Seiberg-Witten equations are perturbed. We introduce three types of SW-invariants and investigate how they are related. In (2.1) we first allow all metrics $g$ and self-dual 2-forms $h$. Then the SW-invariant $SW_X(L)$ defined in (2.2) above has generically two values which are determined by the sign of $(2\pi c_1(L) + [h]) \cdot [\omega_g]$, where $\omega_g$ is the unique $g$-self-dual harmonic 2-form of norm one lying in the (preassigned) positive component of $H^2_{+_g}(X; \mathbf{R})$. We denote the SW-invariant for the metric $g$ and generic self-dual 2-form $h$ satisfying $(2\pi c_1(L) + [h]) \cdot [\omega_g] > 0$ by $SW_X^+(L)$ and denote the other one by $SW_X^-(L)$. Secondly one may perturb the Seiberg-Witten equations by adding only a small generic self-dual 2-form $h \in \Omega^2_{+_g}(X; \mathbf{R})$, so that one can define the SW-invariants as in (2.2) above. In this case we denote the SW-invariant for a metric $g$ satisfying $(2\pi c_1(L)) \cdot [\omega_g] > 0$ by $SW_X^{\circ,+}(L)$ and we denote the other one by $SW_X^{\circ,-}(L)$. Note that, if it exists, $SW_X^{\circ,\pm}(L) = SW_X^{\pm}(L)$. But it sometimes happens that the sign of $(2\pi c_1(L)) \cdot [\omega_g]$ is the same for all metrics, so that there exists only one SW-invariant obtained by a small generic perturbation of the Seiberg-Witten equations. In such a case we define the SW-invariant of $L$ on $X$ by

$$SW_X^{\circ}(L) := \begin{cases} SW_X^{\circ,+}(L) & \text{if } 2\pi c_1(L) \cdot [\omega_g] > 0 \\ SW_X^{\circ,-}(L) & \text{if } 2\pi c_1(L) \cdot [\omega_g] < 0\,. \end{cases}$$

If $SW_X^{\circ}(L) \neq 0$, we call the corresponding $c_1(L)$ (or $L$) a *SW-basic class* of $X$. Then the Seiberg-Witten invariant of $X$, denoted by $SW_X^{\circ} = \sum SW_X^{\circ}(L) \cdot e^{c_1(L)}$, will also be a diffeomorphism invariant. Furthermore we can extend many results obtained for smooth 4-manifolds with $b_2^+ > 1$ to this case. For example, if $X$ is a simply connected closed smooth 4-manifold with $b_2^+ = 1$ and $b_2^- \leq 9$, then there are only finitely many characteristic line bundles $L$ on $X$ such that $SW_X^{\circ}(L) \neq 0$. Finally we introduce one more type of Seiberg-Witten invariants for $b_2^+ = 1$. Given a fixed cohomology class $[x] \in H^2(X; \mathbf{Z})$ with $[x] \cdot [x] \geq 0$, one may divide the set of metrics and self-dual 2-forms into two classes according to the sign of $proj_{+_g}(2\pi c_1(L) + [h]) \cdot [x]$, where $proj_{+_g}$ is the projection of $\Omega^2(X; \mathbf{R})$ onto the space $H^2_{+_g}(X; \mathbf{R})$ of $g$-self-dual harmonic 2-forms. In this case we denote the SW-invariant for a metric $g$ and a generic self-dual 2-form $h$ satisfying $proj_{+_g}(2\pi c_1(L) + [h]) \cdot [x] > 0$ by $SW_X^{[x],+}(L)$ and we denote the other one by $SW_X^{[x],-}(L)$. R. Fintushel and R. Stern used this type of SW-invariants with $[x] = [T]$ for $b_2^+ = 1$ in [**FS2**]. Note that $SW_X^{[T],\pm}(L) = SW_X^{\pm}(L)$.