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Lectures on Heegaard Floer Homology

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Abstract. These are notes for a lecture series on Heegaard Floer homology. Their aim is to study the surgery long exact sequence for these invariants, which relates the Heegaard Floer homology groups of three-manifolds which differ by surgeries along a knot. We sketch here a proof of this result, and give some of its applications. In fact, the primary application we focus on is the Dehn surgery classification of the unknot.

These are notes for the second lecture course on Heegaard Floer homology in the Clay Mathematics Institute Budapest Summer School in June 2004, taught by the first author. Although some of the topics covered in that course did not make it into these notes (specifically, the discussion of "knot Floer homology" which instead is described in the lecture notes for the first course, cf. [**44**]), the central aim has remained largely the same: we have attempted to give a fairly direct path towards some topological applications of the surgery long exact sequence in Heegaard Floer homology. Specifically, the goal was to sketch with the minimum amount of machinery necessary a proof of the Dehn surgery characterization of the unknot, first established in a collaboration with Peter Kronheimer, Tomasz Mrowka, and the authors. (This problem was first solved in [**29**] using Seiberg-Witten gauge theory, rather than Heegaard Floer homology; the approach outlined here can be found in [**39**].)

In Lecture 1, the surgery exact triangle is stated, and some of its immediate applications are given. In Lecture 2, it is proved. Lecture 3 concerns the maps induced by smooth cobordisms between three-manifolds. This is the lecture containing the fewest technical details – though most of those can be found in [**34**]. In Lecture 4, we show how the exact triangle, together with properties of the maps appearing in it, lead to a proof of the Dehn surgery classification of the unknot.

An attempt has been made to keep the discussion as simple as possible. For example, in these notes we avoid the use of "twisted coefficients". This comes at a price: as a result, we do not develop the necessary machinery required to handle knots with genus one. It is hoped that the reader's interest will be sufficiently piqued to study the original papers to fill in this gap. There are also a number of exercises scattered throughout the text, in topics ranging from homological algebra

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and elementary conformal mapping to low-dimensional topology. The reader is strongly encouraged to think through these exercises; some of the proofs in the text rely on them. At the conclusion of each lecture, there is a discussion on further reading on the material.

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1. Introduction to the surgery exact triangle

The exact triangle is a key calculational tool in Heegaard Floer homology. It relates the Heegaard Floer homology groups of three-manifolds obtained by surgeries along a framed knot in a closed, oriented three-manifold. Before stating the result precisely, we review some aspects of Heegaard Floer homology briefly, and then some of the topological constructions involved.

1.1. Background on Heegaard Floer groups: notation. Recall that Heegaard Floer homology is an Abelian group associated to a three-manifold, equipped with a Spin^c structure $\mathfrak{t} \in \mathrm{Spin}^c(Y)$. It comes in several variants.

Let $(\Sigma, \{\alpha_1, ...\alpha_g\}, \{\beta_1, ..., \beta_g\}, z)$ be a Heegaard diagram for Y, where here $\boldsymbol{\alpha} = \{\alpha_1, ..., \alpha_g\}$ and $\boldsymbol{\beta} = \{\beta_1, ..., \beta_g\}$ are attaching circles for two handlebodies bounded by Σ, and $z \in \Sigma - \alpha_1 - ... - \alpha_g - \beta_1 - ... - \beta_g$ is a reference point.

Form the g-fold symmetric product $\mathrm{Sym}^g(\Sigma)$, and let $\mathbb{T}_\alpha$ and $\mathbb{T}_\beta$ be the tori

$$\mathbb{T}_\alpha = \alpha_1 \times ... \times \alpha_g \quad \text{and} \quad \mathbb{T}_\beta = \beta_1 \times ... \times \beta_g.$$

The simplest version of Heegaard Floer homology is the homology groups of a chain complex generated by the intersection points of $\mathbb{T}_\alpha$ with $\mathbb{T}_\beta$: $\widehat{CF}(Y) = \bigoplus_{\mathbf{x} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta} \mathbb{Z}\mathbf{x}$. This is endowed with a differential

$$\partial \mathbf{x} = \sum_{\mathbf{y} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta} \sum_{\{\phi \in \pi_2(\mathbf{x}, \mathbf{y}) \big| \mu(\phi)=1, n_z(\phi)=0\}} \# \left(\frac{\mathcal{M}(\phi)}{\mathbb{R}} \right) \mathbf{y}.$$

Here, $\pi_2(\mathbf{x}, \mathbf{y})$ denotes the space of homology classes of Whitney disks connecting $\mathbf{x}$ and $\mathbf{y}$ [1], $n_z(\phi)$ denotes the algebraic intersection number of a representative of ϕ with the codimension-two submanifold $\{z\} \times \mathrm{Sym}^{g-1}(\Sigma) \subset \mathrm{Sym}^g(\Sigma)$, $\mathcal{M}(\phi)$ denotes the moduli space of pseudo-holomorphic representatives of ϕ, and $\mu(\phi)$ denotes the expected dimension of that moduli space, its Maslov index. Also, $\# \left(\frac{\mathcal{M}(\phi)}{\mathbb{R}} \right)$ is an appropriately signed count of points in the quotient of $\mathcal{M}(\phi)$ by the natural $\mathbb{R}$ action defined by automorphisms of the domain. To avoid a distracting discussion of signs, we sometimes change to the base ring $\mathbb{Z}/2\mathbb{Z}$, where now this coefficient is simply the parity of the number of points in $\mathcal{M}(\phi)/\mathbb{R}$. The loss of generality coming with this procedure is irrelevant for the topological applications appearing later in these lecture notes.

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[1] In the case where $g(\Sigma) > 2$, we have that $\pi_2(\mathrm{Sym}^g(\Sigma)) \cong \mathbb{Z}$, and hence the distinction between homotopy and homology classes of Whitney disks disappears.

There is an obstruction to connecting $\mathbf{x}$ and $\mathbf{y}$ by a Whitney disk, which leads to a splitting of the above chain complex according to Spin^c structures over Y, induced from a partitioning of $\mathbb{T}_\alpha \cap \mathbb{T}_\beta$ according to Spin^c structures, $\widehat{CF}(Y) = \bigoplus_{\mathfrak{t}\in\mathrm{Spin}^c(Y)} \widehat{CF}(Y,\mathfrak{t})$. The homology groups of $\widehat{CF}(Y,\mathfrak{t})$, $\widehat{HF}(Y,\mathfrak{t})$, are topological invariants of Y and the Spin^c structure $\mathfrak{t}$.

There are other versions of these groups, taking into account more of the homology classes $\phi \in \pi_2(\mathbf{x},\mathbf{y})$. Specifically, we consider the boundary operator

$$\partial \mathbf{x} = \sum_{\mathbf{y}\in\mathbb{T}_\alpha\cap\mathbb{T}_\beta} \sum_{\{\phi\in\pi_2(\mathbf{x},\mathbf{y})\big|\mu(\phi)=1\}} \#\left(\frac{\mathcal{M}(\phi)}{\mathbb{R}}\right)\cdot U^{n_z(\phi)}\mathbf{y},$$

where U is a formal variable. This can be thought of as acting on either the free $\mathbb{Z}[U]$-module generated by intersection points of $\mathbb{T}_\alpha \cap \mathbb{T}_\beta$ ($CF^-(Y,\mathfrak{t})$), or the free $\mathbb{Z}[U,U^{-1}]$-module generated by these same intersection points ($CF^\infty(Y,\mathfrak{t})$), or the module with one copy of $\mathcal{T}^+ = \mathbb{Z}[U,U^{-1}]/U\cdot\mathbb{Z}[U]$ for each intersection point ($CF^+(Y,\mathfrak{t})$). Note also that when the first Betti number of Y, $b_1(Y)$, is nonzero, special "admissible" Heegaard diagrams must be used to ensure the necessary finiteness properties for the sums defining the boundary maps. Once this is done, the homology groups of the chain complexes $HF^-(Y,\mathfrak{t})$, $HF^\infty(Y,\mathfrak{t})$, and $HF^+(Y,\mathfrak{t})$ are topological invariants of Y equipped with its Spin^c structure $\mathfrak{t}$.

For instance, when working with $\widehat{HF}$ and HF^+ for a three-manifold with $b_1(Y) > 0$, we need the following notions.

Definition 1.1. Let $(\Sigma,\boldsymbol{\alpha},\boldsymbol{\beta},z)$ be a pointed Heegaard diagram. The attaching curves divide Σ into a collection of components $\{\mathcal{D}_i\}_{i=1}^n$, one of which contains the distinguished point z. Let $P = \sum_i n_i \cdot \mathcal{D}_i$ be a two-chain in Σ. Its boundary can be written as a sum of subarcs of the α_i and the β_j. The two-chain P is called a *periodic domain* its local multiplicity at z vanishes and if for each i the segments of α_i appear with the same multiplicity. (More informally, we express this condition by saying that the boundary of P can be represented as a sum of the α_i and the β_j.) A Heegaard diagram is said to be *weakly admissible* if all the non-trivial periodic domains have both positive and negative local multiplicities.

Exercise 1.2. Identify the group of periodic domains (where the group law is given by addition of two-chains) with $H_2(Y;\mathbb{Z})$.

Weakly admissible Heegaard diagrams can be found for any three-manifold, and the groups $\widehat{HF}(Y,\mathfrak{t})$ and $HF^+(Y,\mathfrak{t})$ are the homology groups of the chain complexes $\widehat{CF}(Y,\mathfrak{t})$ and $CF^+(Y,\mathfrak{t})$ associated to such a diagram. For more details, and also a stronger notion of admissibility which gives HF^- and HF^∞, see for example Subsection 4.2.2 of [**41**]).

Exercise 1.3. Show that, with coefficients in $\mathbb{F} = \mathbb{Z}/2\mathbb{Z}$, $\widehat{HF}(S^1\times S^2) \cong \mathbb{F}\oplus\mathbb{F}$. Note that there is a also a Heegaard diagram for $S^1\times S^2$ for which $\mathbb{T}_\alpha\cap\mathbb{T}_\beta = \emptyset$ (but of course this diagram is not weakly admissible). *Hint:* draw a genus one Heegaard diagram for $S^2\times S^1$.

Exercise 1.4. Let M be a module over the ring $\mathbb{Z}[U]$. Let M_U denote its localization $M_U = M\otimes_{\mathbb{Z}[U]}\mathbb{Z}[U,U^{-1}]$.