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Ozsváth–Szabó Invariants and Contact Surgery

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Abstract. Let $T \subset S^3$ be a right–handed trefoil, and let $Y_r(T)$ be the closed, oriented 3–manifold obtained by performing rational r–surgery on the 3–sphere S^3 along T. In this paper we explain how to use contact surgery and the contact Ozsváth–Szabó invariants to construct positive, tight contact structures on $Y_r(T)$ for every $r \neq 1$. In particular, we give explicit constructions of positive, tight contact structures on the oriented boundaries of the positive E_6 and E_7 plumbings.

1. Introduction

We shall assume throughout the paper that every 3–manifold is connected, closed and oriented. A *contact structure* on a 3–manifold Y is a 2–dimensional distribution $\xi \subset TY$ given as the kernel of a 1–form $\alpha \in \Omega^1(Y)$ such that $\alpha \wedge d\alpha > 0$ everywhere on Y. The pair (Y, ξ) is a *contact 3–manifold.*

The *standard contact structure* ξ_{st} on $S^3 \subset \mathbb{C}^2$ is the distribution of complex tangent lines

$$\xi_{\mathrm{st}} := TS^3 \cap i \cdot TS^3 \subset TS^3.$$

A contact 3–manifold (Y, ξ) is *overtwisted* if there exists an embedded disk $D^2 \hookrightarrow Y$ such that ξ is tangent to D^2 along its boundary ∂D^2. If there is no such disk, (Y, ξ) is *tight.*

It is known that every coorientable 2–plane field on an orientable 3–manifold is homotopic to a contact structure, so one of the central problems in present–day contact topology is:

(P) Which 3–manifolds carry tight contact structures?

The standard contact 3–sphere (S^3, ξ_{st}) is tight [**1**]. Let $T \subset S^3$ be a right–handed trefoil knot and, for every $r \in \mathbb{Q} \cup \{\infty\}$, denote by $Y_r(T)$ the oriented 3–manifold obtained by performing a rational surgery along T with coefficient r. Then, the oriented 3–manifold $Y_1(T)$ (i.e. the Poincaré homology sphere with orientation the opposite of the standard one) does not carry tight contact structures [**4**].

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Until recently, the two most important methods to deal with problem (P) were Eliashberg's *Legendrian surgery* as used, e.g. by Gompf in [**7**], and the *state traversal* method, developed by Ko Honda and based on Giroux's theory of convex surfaces. The limitations of these two methods come from the fact that Legendrian surgery can only prove tightness of Stein fillable contact structures, while the state traversal becomes too complicated in the absence of suitable incompressible surfaces. For example, both methods fail to deal with problem (P) when Y is either $Y_2(T)$ or $Y_3(T)$, because these Seifert fibered 3–manifolds do not contain vertical incompressible tori, nor do they carry symplectically fillable contact structures [**10, 11**]. As a result, for some time it was posed as an open problem whether $Y_2(T)$ or $Y_3(T)$ carried tight contact structures [**6**].

In this paper we illustrate how the contact Ozsváth–Szabó invariants [**19**] can be effectively combined with contact surgery [**2, 3**] to tackle problem (P). In particular, it follows from Theorem 1 below that $Y_2(T)$ and $Y_3(T)$ do indeed carry tight contact structures. Moreover, it follows from the proof of Theorem 1 that such contact structures can be explicitly described as in Figures 1 and 2 (see Section 2 for the explanation of the notation).

THEOREM 1. *Let* $r \in \mathbb{Q} \cup \{\infty\}$*, and denote by* $Y_r(T)$ *the closed, oriented 3–manifold obtained by performing* r*–surgery on the right–handed trefoil knot* $T \subset S^3$*. Then* $Y_r(T)$ *carries a tight contact structure for every* $r \neq 1$*.*

In proving Theorem 1 we first use contact surgery to define contact structures on $Y_r(T)$ for $r \neq 1$, and then show that the contact Ozsváth–Szabó invariants of those structures do not vanish, implying tightness. During the course of the proof we show that the contact invariants are nontrivial for infinitely many tight, not fillable contact 3–manifolds.

REMARK 2. The reader should be aware that in [**13, 14**] we prove results which are more general than the ones presented here. On the other hand, in this paper we try to keep our presentation at a more expository level by concentrating on just a few illustrative examples. In particular, the arguments given here are somewhat different from, and relatively simpler than, the ones used in [**13, 14**].

2. Contact surgery

Let (Y, ξ) be a contact 3–manifold. A knot $K \subset Y$ is *Legendrian* if K is everywhere tangent to ξ, i.e. $TK \subset \xi$. The framing of a Legendrian knot $K \subset Y$ naturally induced by ξ is called the *contact framing* of K. Given a non–zero rational number $r \in \mathbb{Q}$, one can perform *contact r–surgery* on a contact 3–manifold (Y, ξ) along a Legendrian knot $K \subset Y$ to obtain a new contact 3–manifold (Y', ξ') [**2, 3**]. Here Y' is the 3–manifold obtained by smooth r–surgery along K with respect to the contact framing, while ξ' is constructed by extending ξ from the complement of a suitable regular neighborhood of K as a tight contact structure on the glued–up solid torus. If $r \neq 0$ such an extension always exists, and for $r = \frac{1}{k}$ ($k \in \mathbb{Z}$) it is unique [**9**]. When $r = -1$, the corresponding contact surgery is usually called *Legendrian surgery* along K.

As an illustration of the contact surgery construction, consider the Legendrian link whose front projection is given by the left–hand side of Figure 1 (see e.g. [**8**, Section 11.1] for the description of Legendrian links in terms of their front projections). The coefficients next to each component of the diagram mean that one should perform contact (-1)–surgery along the Legendrian trefoil and $(+1)$–surgery along each of the Legendrian unknots. Since the contact framing of the Legendrian trefoil is $+1$ with respect to the Seifert framing while the contact framing of each Legendrian unknot is -1 (see e.g. [**8**, Section 11.1] for these calculations), converting the contact surgeries into smooth surgeries and applying some Kirby calculus gives the right–hand side of Figure 1. Therefore, the picture represents a contact structure on the oriented 3–manifold $Y_2(T)$. According to [**3**, Proposition 7], a contact $r = \frac{p}{q}$–surgery $(p, q \in \mathbb{N})$ on a Legendrian knot K is equivalent to a contact $\frac{1}{k}$–surgery on K followed by a contact $\frac{p}{q-kp}$–surgery on a Legendrian pushoff of K for any integer $k \in \mathbb{N}$ such that $q - kp < 0$. Moreover, by [**3**, Proposition 3] each contact r–surgery along $K \subset (Y, \xi)$ with $r < 0$ is equivalent to a Legendrian surgery along a Legendrian link $\mathbb{L} = \cup_{i=0}^{m} L_i$. The set of all the Legendrian links $\mathbb{L}$ corresponding to all the possible contact r–surgeries along the Legendrian knot K is determined via a simple algorithm by K and the contact surgery coefficient r. The algorithm is the following. Since $1 - r > 1$, there is a continued fraction expansion

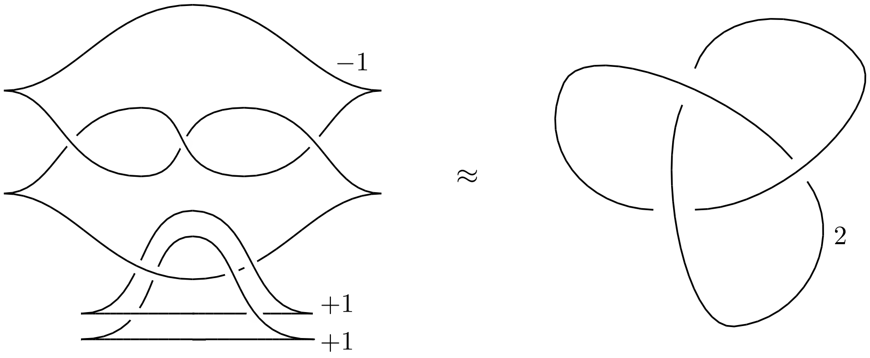

Figure 1. A contact structure on $Y_2(T)$

$$1 - r = a_0 - \cfrac{1}{a_1 - \cfrac{1}{\ddots - \cfrac{1}{a_m}}}, \quad a_0, \ldots, a_m \geq 2.$$

To obtain the first component L_0, push off K using the contact framing and stabilize it $a_0 - 2$ times. Then, push off L_0 and stabilize it $a_1 - 2$ times. Repeat the above scheme for each of the remaining pivots of the continued fraction expansion. Since there are $a_i - 1$ inequivalent ways to stabilize a Legendrian knot $a_i - 2$ times, this construction yields $\Pi_{i=0}^{m}(a_i - 1)$ potentially different Legendrian links.

For example, applying the algorithm just described one can check that the contact surgeries prescribed in the central picture of Figure 2 can be realized in the two ways given by the side pictures of Figure 2. Moreover, converting the coefficients into smooth surgery coefficients and applying Kirby calculus it is easy to check that the underlying 3–manifold is $Y_3(T)$.