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The Kodaira Dimension of Symplectic 4-manifolds

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ABSTRACT. This survey is concerned with the classification of symplectic 4–manifolds. The Kodaira dimension κ divides the symplectic 4–manifolds into 4 classes, each with distinct features. We give an overview of what is known for each class of manifolds.

1. Introduction

Ever since Thurston [**63**] discovered that any T^2-bundle over T^2 with $b_1 = 3$ admits symplectic structures but no Kähler structures, many constructions of closed non-Kähler symplectic $4-$manifolds have appeared. For instance, Gompf [**19**] used the fiber–sum construction to build, for any finitely presented group G, a closed non-Kähler symplectic $4-$manifold M_G with $\pi_1(M_G) = G$ (see also [**3**] for a systematic approach to comparing symplectic 4-manifolds with Kähler surfaces). As a result, it is impossible to classify all symplectic $4-$manifolds. Nevertheless one could attempt to devise a coarse classification scheme. In this regard, the notion of the Kodaira dimension is a perfect place to start.

The Kodaira dimension κ for a Kähler surface is a measure of how positive the canonical bundle is in terms of the growth of plurigenera. The extension of this notion to closed symplectic $4-$manifolds (M, ω) measures the positivity of the *symplectic* canonical class K_ω, and, as for the case of Kähler surfaces, it also takes four values: $-\infty, 0, 1$ and 2. More specifically, for a *minimal* symplectic 4–manifold (M, ω), its Kodaira dimension is defined in terms of the positivity of $K_\omega \cdot [\omega]$ and $K_\omega \cdot K_\omega$. To extend it to general symplectic 4-manifolds, one needs to use results on existence (and uniqueness) of minimal models.

When examined under the lens of the Kodaira dimension, except for T^2-bundles over T^2, all the known non-Kähler symplectic $4-$manifolds have positive values. And the bigger κ is the less we know about the manifolds in that class. The 4–manifolds of $\kappa = -\infty$ have been classified up to symplectomorphisms. There is a conjectured classification for those of $\kappa = 0$. Progress has been made towards bounding the Betti numbers and there is hope of determining their homology types. It is impossible to classify manifolds of positive κ. Instead there are various geography problems and the focus has been on the simply connected ones. We think

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it is also interesting to consider general 4–manifolds of this kind, in particular taking into account the degeneracy and the nullity in the geography problems. The structure of the Gromov-Taubes invariants is also an important problem here.

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2. Definition and basic properties

Let (M,ω) be a closed symplectic 4–manifold. Associated with it is the contractible space of $\omega-$compatible almost complex structures. Thus we can define the symplectic Chern classes $c_i(M,\omega) = c_i(M,J)$ where J is any $\omega-$compatible almost complex structure. In particular, $-c_1(M,\omega) \in H^2(M;\mathbf{Z})$ is called the symplectic canonical class, and is denoted by K_ω.

As mentioned in the introduction, we will first define the Kodaira dimension of (M,ω) when it is minimal, so we need to recall the notion of minimality. Let $\mathcal{E}_M$ be the set of homology classes which have square -1 and are represented by smoothly embedded spheres. M is said to be smoothly minimal if $\mathcal{E}_M$ is empty. Let $\mathcal{E}_{M,\omega}$ be the subset of $\mathcal{E}_M$ which are represented by embedded $\omega-$symplectic spheres. (M,ω) is said to be symplectically minimal if $\mathcal{E}_{M,\omega}$ is empty. When (M,ω) is non-minimal, one can blow down some of the symplectic -1 spheres to obtain a minimal symplectic $4-$manifold (N,μ), which is called a symplectic minimal model of (M,ω) ([**45**]). Now we summarize the basic facts about the minimal models.

Proposition 2.1. *([**28**], [**38**], [**45**], [**60**]) Let M be a closed oriented smooth $4-$manifold and ω a symplectic form on M compatible with the orientation of M.*

1. *M is smoothly minimal if and only if (M,ω) is symplectically minimal. In particular, the underlying smooth manifold of the symplectic minimal model of (M,ω) is smoothly minimal.*
2. *If (M,ω) is not rational nor ruled, then it has a unique symplectic minimal model. Furthermore, for any other symplectic form ω' on M compatible with the orientation of M, the symplectic minimal models of (M,ω) and (M,ω') are diffeomorphic as oriented manifolds.*
3. *If (M,ω) is rational or ruled, then its symplectic minimal models are diffeomorphic to $\mathbf{CP}^2$ or an S^2-bundle over a Riemann surface.*

Here a rational symplectic $4-$manifold is a symplectic $4-$manifold whose underlying smooth manifold is $S^2\times S^2$ or $\mathbf{CP}^2\#k\overline{\mathbf{CP}^2}$ for some non-negative integer k. A ruled symplectic $4-$manifold is a symplectic $4-$manifold whose underlying smooth manifold is the connected sum of a number of (possibly zero) $\overline{\mathbf{CP}^2}$ with an S^2-bundle over a Riemann surface.

Now we are ready to define the symplectic Kodaira dimension.

Definition 2.2. ([**47**], [**30**]) For a minimal symplectic $4-$manifold (M,ω) with symplectic canonical class K_ω, the Kodaira dimension of (M,ω) is defined in the following way:

$$\kappa(M,\omega) = \begin{cases} -\infty & \text{if } K_\omega\cdot[\omega]<0 \text{ or } K_\omega\cdot K_\omega<0,\\ 0 & \text{if } K_\omega\cdot[\omega]=0 \text{ and } K_\omega\cdot K_\omega=0,\\ 1 & \text{if } K_\omega\cdot[\omega]>0 \text{ and } K_\omega\cdot K_\omega=0,\\ 2 & \text{if } K_\omega\cdot[\omega]>0 \text{ and } K_\omega\cdot K_\omega>0.\end{cases}$$

The Kodaira dimension of a non-minimal manifold is defined to be that of any of its symplectic minimal models.

Remark 2.3. In [**47**] the Kodaira dimension of a minimal symplectic $4-$manifold (M, ω) is defined to be $-\infty$ if $K_\omega \cdot [\omega] < 0$, and zero if $K_\omega \cdot [\omega] = 0$. Our modification in [**30**] is to take into account the sign of $K_\omega \cdot K_\omega$ as well in these two cases. Since, for any minimal ruled surface with negative $K_\omega \cdot K_\omega$, there are symplectic forms ω with $K_\omega \cdot [\omega]$ non-negative, this slight modification is necessary for the Kodaira dimension to be well-defined for all symplectic $4-$manifolds.

For a minimal symplectic $4-$manifold, its Kodaira dimension has the following properties.

Theorem 2.4. *Let M be a closed oriented smooth $4-$manifold and ω a symplectic form on M compatible with the orientation of M. If (M, ω) is symplectically minimal, then*

(1) *The Kodaira dimension of (M, ω) is well-defined.*
(2) *(M, ω) has Kodaira dimension $-\infty$ if and only if it is rational or ruled.*
(3) *(M, ω) has Kodaira dimension 0 if and only if K_ω is a torsion class.*

Furthermore $\kappa(M, \omega)$ is well-defined for any symplectic 4-manifold (M, ω).

All the properties are based on the Taubes-Seiberg-Witten theory (cf. [**59**], [**60**] and [**38**]). To show (1) amounts to showing that any minimal symplectic 4-manifold must satisfy one and only one of the four conditions above, i.e. there is no minimal manifold (M, ω) with $K_\omega \cdot K_\omega > 0$ and $K_\omega \cdot [\omega] = 0$. This is an immediate consequence of the following fact proved in [**30**]: If (M, ω) is minimal with $K_\omega \cdot [\omega] = 0$ and $K_\omega \cdot K_\omega \geq 0$, then K_ω is a torsion class and hence $K_\omega \cdot K_\omega = 0$. Notice that property (3) also follows from this fact. Property (2) follows from [**41**]. The last property is now a consequence of Proposition 2.1. If (M, ω) is not rational or ruled, it has a unique symplectic minimal model by Proposition 2.1 (2), so $\kappa(M, \omega)$ is well defined by property (1). If (M, ω) is rational or ruled, it has non-diffeomorphic symplectic minimal models. However the different minimal models are still rational or ruled by Proposition 2.1 (3), so all have Kodaira dimension $-\infty$ by property (2).

There are two additional properties for $\kappa(M, \omega)$. It is not hard to verify that the holomorphic Kodaira dimension of a Kähler surface coincides with the Kodaira dimension of the underlying symplectic 4-manifold. Furthermore, the Kodaira dimension of (M, ω) only depends on the oriented diffeomorphism type of M, i.e. if ω' is another symplectic form on M compatible with the orientation of M, then $\kappa(M, \omega) = \kappa(M, \omega')$.

Remark 2.5. We would like to see whether it is possible to define $\kappa(M, \omega)$ for higher dimensional symplectic manifolds. Again we would first define it for 'minimal' manifolds of dimension $2n$ as follows: $\kappa(M, \omega)$ is defined to be $-\infty$ if $K_\omega^i \cdot [\omega]^{n-i}$ is negative for some i; and $\kappa(M, \omega) = i$ if $K_\omega^j \cdot [\omega]^{n-j} = 0$ for any $j \geq i+1$ and $K_\omega^j \cdot [\omega]^{n-j} > 0$ for any $j < i+1$. To show it is well-defined we need to exclude other possibilities of the $n+1$ numbers $\{K_\omega^i [\omega]^{n-i}\}_{i=0}^n$. Then we would extend it to general manifolds by requiring 'birational' invariance. Of course this is just a speculation since there are many issues to be settled here, one of which is that different minimal models of a manifold should have the same Kodaira dimension.