Introduction

The Clay Mathematical Institute hosted its 2004 Summer School on *Floer homology, gauge theory, and low–dimensional topology* at the Alfréd Rényi Institute of Mathematics in Budapest, Hungary. The aim of this school was to bring together students and researchers in the rapidly developing crossroads of gauge theory and low–dimensional topology. In part, the hope was to foster dialogue across closely related disciplines, some of which were developing in relative isolation until fairly recently. The lectures centered on several topics, including Heegaard Floer theory, knot theory, symplectic and contact topology, and Seiberg–Witten theory. This volume is based on lecture notes from the school, some of which were written in close collaboration with assigned teaching assistants. The lectures have revised the choice of material somewhat from that presented at the school, and the topics have been organized to fit together in logical categories. Each course consisted of two to five lectures, and some had associated problem sessions in the afternoons.

Mathematical gauge theory studies connections on principal bundles, or, more precisely, the solution spaces of certain partial differential equations for such connections. Historically, these equations have come from mathematical physics. Gauge theory as a tool for studying topological properties of four–manifolds was pioneered by the fundamental work of Simon Donaldson in the early 1980's. Since the birth of the subject, it has retained its close connection with symplectic topology, a subject whose intricate structure was illuminated by Mikhail Gromov's introduction of pseudo–holomorphic curve techniques, also introduced in the early 1980's. The analogy between these two fields of study was further underscored by Andreas Floer's construction of an infinite–dimensional variant of Morse theory that applies in two *a priori* different contexts: either to define symplectic invariants for pairs of Lagrangian submanifolds of a symplectic manifold (the so–called *Lagrangian Floer homology*), providing obstructions to disjoining the submanifolds through Hamiltonian isotopies, or to give topological invariants for three–manifolds (the so–called *instanton Floer homology*), which fit into a framework for calculating Donaldson's invariants for smooth four–manifolds.

In the mid–1990's, gauge–theoretic invariants for four–manifolds underwent a dramatic change with the introduction of a new set of partial differential equations introduced by Nathan Seiberg and Edward Witten in their study of string theory. Very closely connected with the underlying geometry of the four–manifolds over which they are defined, the Seiberg–Witten equations lead to four–manifold invariants which are in many ways much easier to work with than the anti–self–dual Yang–Mills equations which Donaldson had studied. The introduction of the new invariants led to a revolution in the field of smooth four–manifold topology.

Highlights in four–manifold topology from this period include the deep theorems of Clifford Taubes about the differential topology of symplectic four–manifolds. These give an interpretation of some of Gromov's invariants for symplectic manifolds in terms of the Seiberg–Witten invariants of the underlying smooth four–manifold. Another striking consequence of the new invariants was a quick, elegant proof by Kronheimer and Mrowka of a conjecture by Thom, stating that the algebraic curves in the complex projective plane minimize genus in their homology class. The invariants were also used particularly effectively in work of Ron Fintushel and Ron Stern, who discovered several operations on smooth four–manifolds, for which the Seiberg–Witten invariants transform in a predictable manner. These operations include *rational blow–downs*, where the neighborhood of a certain chain of spheres is replaced by a space with vanishing second homology, and also *knot surgery*, for which the Alexander polynomial of a knot is reflected in the Seiberg–Witten invariants of a corresponding four–manifold. These operations can be used to construct a number of smooth four–manifolds with interesting properties.

In an attempt to better understand the somewhat elusive gauge theoretic invariants, a different construction was given by Peter Ozsváth and Zoltán Szabó. They formulated an invariant for three– and four–manifolds which takes as its starting point a Lagrangian Floer homology associated to Heegaard diagrams for three–manifolds. The resulting "Heegaard Floer homology" theory is conjecturally isomorphic to Seiberg–Witten theory, but more topological and combinatorial in its flavor and correspondingly easier to work with in certain contexts. Moreover, this theory has benefitted a great deal from an array of contemporary results rendering various analytical and geometric structures in a more topological and combinatorial form, such as Donaldson's introduction of "Lefschetz pencils" in the symplectic category and Giroux's correspondence between open book decompositions and contact structures.

The two lecture series of Ozsváth and Szabó in the first section of this volume provide a leisurely introduction to Heegaard Floer theory. The first lecture series (the lectures given by Szabó at the Summer School) start with the basic notions, and move on to the constructions of the primary variants of Floer homology groups and maps between them. These lectures also cover basics of a corresponding Heegaard Floer homology invariant for knots. The second lecture series (given by Ozsváth) gives a rapid proof of one of the basic calculational tools of the subject, the surgery exact triangle, and its immediate applications. Special emphasis is placed on a Dehn surgery characterization of the unknot, a result whose proof is outlined in these lectures. Section 1 concludes with the lecture notes from Goda's course. Whereas Heegaard diagrams correspond to real–valued Morse theory in three dimensions, in these lectures, Goda considers circle–valued Morse theory for link complements. He uses this theory to give obstructions to a knot being fibered.

The main theme in Section 2 is contact geometry and its interplay with Floer homology. The lectures of John Etnyre give a detailed account of open book decompositions and contact structures, and the Giroux correspondence. The proof of the Giroux correspondence is followed by some applications of this theory, including an embedding theorem for weak symplectic fillings, which turned out to be a crucial step in many of the recent developments of the subject, including the verification of *Property P* by Kronheimer and Mrowka. The definition of the contact invariant in Heegaard Floer theory (resting on the above mentioned Giroux correspondence) is discussed in the lecture notes of András Stipsicz, together with a short discussion on contact surgeries. Results regarding existence of tight contact structures on various 3–manifolds and their fillability properties are also given. A similar application of the contact invariants is described in the paper of Paolo Lisca and András Stipsicz, with the use of minimum machinery required in the proof. A different type of Floer homology (called *contact homology*) is studied in Tobias Ekholm's paper. A classical result of Gromov states that any exact Lagrangian immersion into $\mathbb{C}^n$ has at least one double–point. Ekholm generalizes this result, using Floer homology to give estimates on the minimum number of double–points of an exact Legendrian immersion into some Euclidean space.

Section 3 discusses symplectic geometry and Seiberg–Witten invariants. Ron Fintushel's lectures give an introduction to Seiberg–Witten invariants and the knot surgery construction. The lectures give a thorough discussion of how the Seiberg–Witten invariants transform under the knot surgery operation. Applications include exotic embeddings of surfaces in smooth four–manifolds. Ron Stern's contribution describes the current state of art in the classification of smooth 4–manifolds, and collects a number of intriguing questions and problems which can motivate further results in the subject. The paper of Jongil Park provides new applications of the rational blow–down construction, which led him to discover symplectic 4–manifolds homeomorphic but not diffeomorphic to rational surfaces with small Euler characteristic. Tian–Jun Li studies symplectic 4–manifolds systematically using the generalization of the notion of the holomorphic Kodaira dimension κ to this category. After the discussion of the $\kappa = -\infty$ case, the state of the art for $\kappa = 0$ is described, where a reasonably nice classification scheme is expected. The contribution of Denis Auroux also addresses the problem of understanding symplectic 4–manifolds, but from a completely different point of view. In this case the manifolds are presented as branched covers of the complex projective plane along certain curves, and the discussion centers on the possibility of getting symplectic invariants from topological properties of these branch sets. The volume concludes with Ivan Smith's contribution, where the author reviews basics about symplectic fibrations, leading him (in a joint project with Paul Seidel) to knot invariants defined using symplectic topology and Floer homology, conjecturally recapturing the celebrated knot invariants of Khovanov.

It is hoped that this volume will give the reader a sampling of these many new and exciting developments in low–dimensional topology and symplectic geometry. Before commencing with the mathematics, we would like to pause to thank some of the many people who have contributed in one way or another to this volume. We would like to thank Arthur Greenspoon for a meticulous proofreading of this text. We would like to thank the Clay Mathematical Institute for making this program possible, through both their financial support and their enthusiasm; special mention goes to Vida Salahi for her careful and diligent work in bringing this volume to print. Next, we thank the staff at the Rényi Institute for helping to create a conducive environment for the Summer School. We would like to thank the lecturers for giving clear, accessible accounts of their research, and we are also grateful to their course assistants, who helped make these courses run smoothly. Finally, we thank

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David Ellwood, Peter Ozsváth, András Stipsicz, Zoltán Szabó

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