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Circle Valued Morse Theory for Knots and Links

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Abstract. We apply a circle valued Morse map to the complements of knots and links in the 3-sphere, and observe their topology including the (twisted) Alexander polynomial, Novikov homology, and two types of Reidemeister torsion.

1. Introduction

Let M be a smooth manifold. Traditional Morse theory deals with a real-valued function $f : M \to \mathbb{R}$. This function corresponds to a handle decomposition of M via Morse's lemma giving Morse's inequality. It describes the relationship between the number of critical points of f and the Betti number and the torsion number of M. The critical points of a Morse function f generate the Morse-Smale complex $C^{MS}(f)$ over $\mathbb{Z}$, using the gradient flow to define the differentials. It is easy to see Morse's inequality from the isomorphism $H_*(C^{MS}(f)) \cong H_*(M)$.

The more recent Morse theory of a circle valued Morse map $f : M \to S^1$ is more complicated, but shares many features of the real valued theory. As in the case of a real valued Morse theory, we have an inequality, which is called the Morse-Novikov inequality, and then the critical points of a circle valued Morse map f generate the Novikov complex $C^{\mathrm{nov}}(f)$ over the Novikov ring $\mathbb{Z}((z))$ of formal power series with integer coefficients, using the gradient flow of the real valued Morse function $\overline{f} : \overline{M} \to \mathbb{R}$ on the infinite cyclic cover to define the differentials. The Novikov homology is the $\mathbb{Z}((z))$-coefficient homology of $\overline{M}$. This theory was started by Novikov [**31**]. See [**34**] for a survey of these topics.

Recently, there are some works on the circle valued Morse theory for the complement of a knot or link in the 3-sphere S^3. We focus on it in this paper, and give a survey.

We define a circle valued Morse map with some conditions on the complement of a link L in S^3, and the *Morse-Novikov number* $\mathcal{MN}(L)$, i.e., the minimal possible number of critical points, roughly speaking. In particular, a link L is fibred if and only if $\mathcal{MN}(L) = 0$. We observe some properties of $\mathcal{MN}(L)$ in Section 2. There is a handle decomposition which corresponds to this Morse map, which is called

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Heegaard splitting for sutured manifolds. We introduce this notion in Section 3, and give some concrete examples. Furthermore we consider the behavior of $\mathcal{MN}(L)$.

By using Alexander ideals (polynomials), we describe a Morse-type inequality, originally due to Pajitnov, Rudolph and Weber in [**33**]. In Section 4, we review their theorem and see some examples. Note that this theorem may be regarded as an extension of the results of Neuwirth [**30**] and Stallings [**38**]. In Section 5, we generalize the theorem of Neuwirth and Stallings using the twisted Alexander invariant which was defined by Wada [**39**]. This observation leads to the definition of *twisted Novikov homology*. We present the definition in Section 6, and show an estimate which generalizes the theorem of Pajitnov, Rudolph and Weber.

Hutchings and Lee showed the relationship between Reidemeister torsion of an ordinary complex and that of a Novikov complex in [**17**], [**18**]. In Section 7, we observe this result through some calculations of both torsions for some 3-manifolds. Mark's work [**26**] plays an important role here.

Terminology and notation. Throughout this paper, we work in the C^∞-category. Thus, the functions, maps, curves, etc. are assumed to be of class C^∞.

Let L be an oriented link in S^3, and let $C_L = S^3 - L$. Further, let $E(L) = S^3 - \operatorname{Int} N(L)$ be its exterior, where $N(L)$ is a regular neighborhood of L in S^3.

A *Seifert surface* is an oriented compact 2-submanifold of S^3 with no closed component. The boundary $L = \partial\bar{R}$ of a Seifert surface $\bar{R}$ is an oriented link; $\bar{R}$ is called a *Seifert surface for* L. The intersection of $\bar{R}$ with $E(L)$, $R = \bar{R} \cap E(L)$, is also called a Seifert surface for L.

2. Circle valued Morse map for knots and links

In this section, we review some definitions and the basic properties of circle valued Morse maps for knots and links.

A Morse map $f : C_L \to S^1$ is said to be *regular* if each component L_i of L has a neighborhood framed as $S^1 \times D^2$ such that $L_i = S^1 \times \{0\}$ and the restriction $f| : S^1 \times (D^2 - \{0\}) \to S^1$ is given by $(x, y) \to y/|y|$. Let $m_p(f)$ be the number of critical points of f of index p. We say that a Morse map $f : C_L \to S^1$ is *minimal* if it is regular and for each p, $m_p(f)$ is minimal possible among all regular maps homotopic to f. Suppose f is minimal. We call $\mathcal{MN}(L) = \sum_p m_p(f)$ the *Morse-Novikov number* of L. Note that even in the case of a real valued Morse function on a manifold M, minimal Morse functions do not always exist. The problem is that, in general, $m_p(f)$ cannot be minimized for all indices p simultaneously. However, in [**33**], Pajitnov, Rudolph and Weber show that in the case where $M = C_L$ a minimal Morse map exists with some nice properties.

Definition 2.1. A regular Morse map $f : C_L \to S^1$ is said to be *moderate* if

(i) $m_0(f) = m_3(f) = 0$,
(ii) all critical values corresponding to critical points of the same index coincide,
(iii) $f^{-1}(x)$ is a connected Seifert surface for any regular value $x \in S^1$.

Theorem 2.2 ([**33**]). *Every link has a minimal Morse map which is moderate.*

From this theorem we have:

Corollary 2.3. (1) Let f be a moderate map; then $m_1(f) = m_2(f)$.

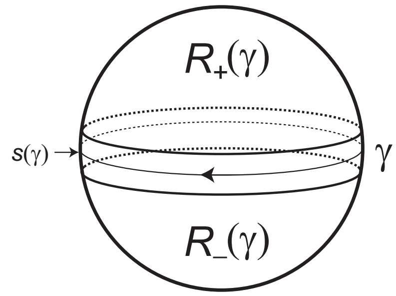

FIGURE 1. The trivial sutured manifold

(2) Let f be a regular Morse map realizing $\mathcal{MN}(L)$; then we may suppose $\mathcal{MN}(L) = m_1(f) + m_2(f)$.

3. Heegaard splitting for sutured manifolds

3.1. Definition. The concept of sutured manifold was defined by Gabai [**7**]. It is a very useful tool in studying knots and links. Here we present an application.

First, we define a sutured manifold in our setting.

DEFINITION 3.1 (sutured manifold). A sutured manifold (M, γ) is a compact oriented 3-manifold M together with a subset $\gamma \subset \partial M$ which is a union of finitely many mutually disjoint annuli. For each component of γ, a *suture*, that is, an oriented core circle, is fixed, and $s(\gamma)$ denotes the set of sutures. Every component of $R(\gamma) = \partial M - \text{Int}\,\gamma$ is oriented so that the orientations on $R(\gamma)$ are coherent with respect to $s(\gamma)$, i.e., the orientation of each component of $\partial R(\gamma)$, which is induced by that of $R(\gamma)$, is parallel to the orientation of the corresponding component of $s(\gamma)$. Let $R_+(\gamma)$ (resp. $R_-(\gamma)$) denotes the union of those components of $R(\gamma)$ whose normal vectors point out of (resp. into) M. In the case that (M, γ) is homeomorphic to $(F \times [0, 1], \partial F \times [0, 1])$ where F is a compact oriented 2-manifold, (M, γ) is called a *product sutured manifold.*

Let L be an oriented link in S^3, and $\bar{R}$ a Seifert surface of L. Set $R = \bar{R} \cap E(L)$ $(E(L) = \text{cl}(S^3 - N(L)))$, and $(P, \delta) = (N(R, E(L)), N(\partial R, \partial E(L)))$. We call (P, δ) a *product sutured manifold for* R. Let $(M, \gamma) = (\text{cl}(E(L) - P), \text{cl}(\partial E(L) - \delta))$ with $R_\pm(\gamma) = R_\mp(\delta)$. We call (M, γ) a *complementary sutured manifold* for R. In this paper, we call this a *sutured manifold* for short.

Let (V, γ) be a sutured manifold such that V is a 3-ball and γ is an annulus embedded in ∂V. Then, we call (V, γ) the *trivial sutured manifold*. See Figure 1.

EXAMPLE 3.2. Let K be the trivial knot. Then K has a Seifert surface D that is a disk. The product sutured manifold for D is the trivial sutured manifold. Further, the (complementary) sutured manifold for D is also the trivial sutured manifold.

EXAMPLE 3.3. The left hand side figure in Figure 2 is the trefoil knot K, and the middle is a Seifert surface R of K. The (complementary) sutured manifold for R is homeomorphic to the manifold the right hand side of the figure. (Note that the 'outside' of the genus 2 surface is the complementary sutured manifold.)