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Knot Surgery Revisited

Ronald Fintushel

Abstract. We give an introduction to the topology of smooth 4-manifolds by studying three different proofs of the "knot surgery theorem".

Introduction

This survey is comprised of lectures given at the 2004 Clay Mathematics Institute Summer School in Budapest. My task was to give a general introduction to 4-manifolds in five lectures. (A paraphrasing of this might have been a more clever title for this article.) Since the stated goal seemed to me to be impossible, I instead tried to concentrate on one theorem — relating the Seiberg-Witten invariant of the result of knot surgery to the Alexander polynomial. This theorem has had several proofs from different points of view, and I thought that talking about them would give a nice overview of some of the techniques used in 4-manifold theory.

This article begins with a section which gives a 'user's guide' to Seiberg-Witten theory, concentrating on gluing theorems. Section 2 describes knot surgery and some simple applications. It then outlines the proof due to Ron Stern and myself of the knot surgery theorem: that knot surgery with a knot K has the effect of multiplying the Seiberg-Witten invariant by the Alexander polynomial of K. This proof is based on the relationship of the 'macareña' technique for calculating the Alexander polynomial with surgery formulas for the Seiberg-Witten invariant.

The knot surgery theorem is closely related to the Meng-Taubes Theorem, which relates the Seiberg-Witten invariant of a 3-manifold to its Milnor torsion. This theorem and its relationship to knot surgery is discussed in Section 3, where we give an introduction to the beautiful paper [**D**] of Simon Donaldson. Donaldson's proof relates the Seiberg-Witten invariant of a 3-manifold Y which has the homology of $S^2 \times S^1$ to the abelian vortex equations on a Riemann surface using ideas from topological quantum field theory. Our notes cover the case where Y is fibered over the circle. (There is also a nice exposition of this in unpublished notes of Ivan Smith.)

In Section 4, we have given a short introduction to the Taubes-Gromov theory approach to calculating Seiberg-Witten invariants of symplectic 4-manifolds. After some general comments concerning the definition of Gromov invariants and Taubes'

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theorem on the their equivalence to Seiberg-Witten invariants, we discuss a proof of the Meng-Taubes formula from this point of view following one given by Taubes in [**T3**].

In the final section we discuss joint work with Ron Stern which applies knot surgery to the problem of constructing exotic embedded surfaces in 4-manifolds. The two techniques which are covered are 'rim surgery' which allows the exotic reimbedding of smooth surfaces in a fixed topological type (X, Σ), and braiding, which allows the construction of exotic symplectic tori in a fixed homology class.

I hope that no one will misconstrue this survey as being definitive in any sense. One can always learn more by going back to the papers that I have cited. If these notes or my lectures have convinced anyone to do that, they will have more than served their purpose.

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1. Seiberg-Witten Invariants and Gluing

The main tool used to understand smooth structures on 4-manifolds is the Seiberg-Witten invariant. The goal of this lecture is to provide a 'user's guide' to these invariants. For more detailed explanations one should see [**W, KM, M, N**].

Consider a smooth compact oriented 4-manifold X with tangent bundle TX. The choice of a Riemannian metric on X reduces the structure group of TX to $\mathrm{SO}(4)$, which may be equivalently taken as the structure group of PX, the bundle of tangent frames of X. The double covering group of $\mathrm{SO}(4)$ is $\mathrm{Spin}(4) \cong \mathrm{SU}(2) \times \mathrm{SU}(2)$, and a spin-structure on X is a lift of PX to a principal $\mathrm{Spin}(4)$-bundle $\tilde{P}X$ over X such that in the diagram

$$\begin{tikzcd}
\tilde{P}X \arrow[rr] \arrow[dr] & & PX \arrow[dl] \\
& X &
\end{tikzcd}$$

the horizontal map is a double cover on each fiber of PX.

A spin structure gives rise to spinor bundles $S^{\pm} = \tilde{P}X \times_{\mathrm{SU}(2)} \mathbf{C}^2$, where the action of $\mathrm{SU}(2)$ on $\tilde{P}X$ arises from one of the two factors of $\mathrm{Spin}(4) \cong \mathrm{SU}(2) \times \mathrm{SU}(2)$. From the point of view of algebraic topology, one can think of a spin structure on X as a lift

$$\begin{tikzcd}
& B\mathrm{Spin}(4) \arrow[d] \\
X \arrow[r] \arrow[ur, dotted] & B\mathrm{SO}(4)
\end{tikzcd}$$

The obstruction to finding such a lift is the second Stiefel-Whitney class $w_2(X) \in H_2(X; \mathbf{Z}_2)$. One may alternatively think in terms of the transition functions

$$\{\varphi_{i,j} : U_i \cap U_j \to \mathrm{SO}(4)\}$$

of PX. A spin structure on X consists of lifts $\tilde{\varphi}_{i,j} : U_i \cap U_j \to \mathrm{Spin}(4)$. In order to give a bundle $\tilde{P}X$, these lifts must satisfy the cocycle condition $\tilde{\varphi}_{i,j} \circ \tilde{\varphi}_{j,k} = \tilde{\varphi}_{i,k}$. From this point of view, $\tilde{P}X$ corresponds to an element $\tilde{\xi}$ of the Čech cohomology group $H^1(X; \mathrm{Spin}(4))$ such that in the sequence

$$\cdots \to H^1(X; \mathbf{Z}_2) \xrightarrow{i_*} H^1(X; \mathrm{Spin}(4)) \xrightarrow{p_*} H^1(X; \mathrm{SO}(4)) \xrightarrow{\delta} H^2(X; \mathbf{Z}_2) \to \dots$$

$p_*\tilde{\xi} = \xi$, the element which corresponds to PX. Note that $\delta\xi = w_2(X)$, affirming our comment above. Also note that if X admits a spin structure (i.e. a lift of ξ), such lifts are in 1-1 correspondence with $H^1(X; \mathbf{Z}_2)$. To each spin structure there is associated a Dirac operator $D : \Gamma(S^+) \to \Gamma(S^-)$, an elliptic operator which plays an important role in topology and geometry.

In case $w_2(X) \neq 0$, X admits no spin structure, but it can still admit a spin$^{\mathrm{c}}$ -structure. A spin$^{\mathrm{c}}$ structure is given by a pair of rank 2 complex vector bundles $W^{\pm}$ over X with isomorphisms $\det(W^+) = \det(W^-) = L$, some complex line bundle over X, so that locally $W^{\pm} = S^{\pm} \otimes L^{\frac{1}{2}}$. To make sense of this, consider the transition maps $\{\varphi_{i,j} : U_i \cap U_J \to \mathrm{SO}(4)\}$ for PX. We can assume that our charts have overlaps $U_i \cap U_J$ which are contractible, so that we can always get lifts $\tilde{\varphi}_{i,j} : U_i \cap U_J \to \mathrm{Spin}(4)$. However, if $w_2(X) \neq 0$, we can never find lifts satisfying the cocycle condition.

Similarly, suppose that we are given a complex line bundle L with transition functions $\{g_{i,j} : U_i \cap U_j \to \mathrm{U}(1)\}$. Locally these functions have square roots $(g_{i,j})^{\frac{1}{2}}$. The obstruction to finding a system of square roots which satisfy the cocycle condition, i.e. to finding a global bundle $L^{\frac{1}{2}}$ over X such that $L^{\frac{1}{2}} \otimes L^{\frac{1}{2}} \cong L$ is $c_1(L) \pmod 2$ in $H^2(X; \mathbf{Z}_2)$. Now suppose that L is characteristic, i.e. that $w_2(X) = c_1(L) \pmod 2$. The statement that $W^{\pm}$ should locally be $S^{\pm} \otimes L^{\frac{1}{2}}$ means that the tensor products $\tilde{\varphi}_{i,j} \otimes (g_{i,j})^{\frac{1}{2}}$ should satisfy the cocycle condition. This function has values in $(\mathrm{U}(1) \times \mathrm{SU}(2) \times \mathrm{SU}(2))/\{\pm 1\} = \mathrm{Spin}^{\mathrm{c}}(4)$, and the corresponding obstruction is $2w_2(X) = 0$; so spin$^{\mathrm{c}}$ structures exist provided we can find characteristic line bundles L over X. A theorem of Hirzebruch and Hopf states that these exist on any oriented 4-manifold [**HH**]. Spin$^{\mathrm{c}}$ structures on X are classified by lifts of $w_2(X)$ to $H^2(X; \mathbf{Z})$ up to the action of $H^1(X; \mathbf{Z}_2)$. (Spin structures correspond to $0 \in H^2(X, \mathbf{Z})$ up to this action.)

The group $\mathrm{Spin}^{\mathrm{c}}(4) \cong (\mathrm{U}(1) \times \mathrm{SU}(2) \times \mathrm{SU}(2))/\{\pm 1\}$ fibers over $SO(4) \cong (\mathrm{SU}(2) \times \mathrm{SU}(2))/\{\pm 1\}$ with fiber $S^1 \cong \mathrm{U}(1)$. A spin$^{\mathrm{c}}$ structure s on X is a lift of PX to a principal $\mathrm{Spin}^{\mathrm{c}}(4)$ bundle $\hat{P}_X$ over X. Since $\mathrm{U}(2) \cong (\mathrm{U}(1) \times \mathrm{SU}(2))/\{\pm 1\}$, we get representations $s^{\pm} : \mathrm{Spin}^{\mathrm{c}}(4) \to \mathrm{U}(2)$, and associated rank 2 complex vector bundles

$$W^{\pm} = \hat{P}_X \times_{s^{\pm}} \mathbf{C}^2$$

called spinor bundles, and referred to above, and $L = \det(W^{\pm})$. We sometimes write $c_1(s)$ for $c_1(L)$.

As for ordinary spin structures, one has Clifford multiplication

$$c : T^*X \otimes W^{\pm} \to W^{\mp}$$

written $c(v, w) = v.w$ and satisfying $v.(v.w) = -|v|^2 w$. Thus c induces a map

$$c : T^*X \to \mathrm{Hom}(W^+, W^-)$$

A connection A on L together with the Levi-Civita connection on the tangent bundle of X forms a connection $\nabla_A : \Gamma(W^+) \to \Gamma(T^*X \otimes W^+)$ on W^+. This