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Lectures on Open Book Decompositions and Contact Structures

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ABSTRACT. This article provides a brief introduction to open book decompositions of 3-manifolds and sketches the proof of Giroux's correspondence between these open books and oriented contact structures on closed 3-manifolds. We then discuss applications of this correspondence to symplectic fillings. This circle of ideas has been essential to recent progress in contact geometry and applications of Heegaard Floer homology and gauge theory to low-dimensional topology.

1. Introduction

The main goal of this survey is to discuss the proof and examine some consequences of the following fundamental theorem of Giroux.

THEOREM 1.1 (Giroux 2000, [**21**]). *Let M be a closed oriented 3-manifold. Then there is a one-to-one correspondence between*

$$\{\textit{oriented contact structures on } M \textit{ up to isotopy}\}$$

and

$$\{\textit{open book decompositions of } M \textit{ up to positive stabilization}\}.$$

This theorem plays a pivotal role in studying cobordisms of contact structures and understanding filling properties of contact structures, see [**2, 6, 13, 14, 16, 19**]. This better understanding of fillings leads to various topological applications of contact geometry. Specifically, the much studied property P for knots was established by P. Kronheimer and T. Mrowka in [**28**]. A non-trivial knot has property P if non-trivial surgery on it never gives a homotopy sphere. In addition P. Ozsváth and Z. Szabó in [**34**] gave an alternate proof of a characterization of the unknot via surgery which was originally established in [**29**]. This characterization says that the unknot is the only knot on which p-surgery yields $-L(p,1)$. Moreover, in [**34**] it is shown that the Thurston norm is determined by Heegaard Floer Homology.

Ideally the reader should be familiar with low-dimensional topology at the level of, say [**36**]. In particular, we will assume familiarity with Dehn surgery, mapping tori and basic algebraic topology. At various points we also discuss branch coverings, Heegaard splittings and other notions; however, the reader unfamiliar with these

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notions should be able to skim these parts of the paper without missing much, if any, of the main line of the arguments. Since diffeomorphisms of surfaces play a central role in much of the paper and specific conventions are important we have included an Appendix discussing basic facts about this. We also assume the reader has some familiarity with contact geometry. Having read [**15**] should be sufficient background for this paper. In order to accommodate the reader with little background in contact geometry we have included brief discussions, scattered throughout the paper, of all the necessary facts. Other good introductions to contact geometry are [**1, 20**], though a basic understanding of convex surfaces is also useful but is not covered in these sources.

In the next three sections we give a thorough sketch of the proof of Theorem 1.1. In Section 2 we define open book decompositions of 3-manifolds, discuss their existence and various constructions. The following two sections discuss how to get a contact structure from an open book and an open book from a contact structure, respectively. Finally in Section 5 we will consider various applications of Theorem 1.1. While we prove various things about open books and contact structures our main goal is to prove the following theorem which is the basis for most of the above mentioned applications of contact geometry to topology.

THEOREM (Eliashberg 2004 [**6**]; Etnyre 2004 [**14**]). *If* (X, ω) *is a symplectic filling of* (M, ξ) *then there is a closed symplectic manifold* (W, ω') *and a symplectic embedding* $(X, \omega) \to (W, \omega')$.

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2. Open book decompositions of 3-manifolds

Throughout this section (and these notes)

M is always a closed oriented 3-manifold.

We also mention that when inducing an orientation on the boundary of a manifold we use the "outward normal first" convention. That is, given an oriented manifold N then $v_1, \ldots, v_{n-1}$ is an oriented basis for ∂N if $\nu, v_1, \ldots, v_{n-1}$ is an oriented basis for N.

DEFINITION 2.1. An *open book decomposition of* M is a pair (B, π) where

(1) B is an oriented link in M called the *binding* of the open book and
(2) $\pi : M \setminus B \to S^1$ is a fibration of the complement of B such that $\pi^{-1}(\theta)$ is the interior of a compact surface $\Sigma_\theta \subset M$ and $\partial \Sigma_\theta = B$ for all $\theta \in S^1$. The surface $\Sigma = \Sigma_\theta$, for any θ, is called the *page* of the open book.

One should note that it is important to include the projection in the data for an open book, since B does not determine the open book, as the following example shows.

EXAMPLE 2.2. Let $M = S^1 \times S^2$ and $B = S^1 \times \{N, S\}$, where $N, S \in S^2$. There are many ways to fiber $M \setminus B = S^1 \times S^1 \times (0,1)$. In particular if γ_n is an embedded curve on T^2 in the homology class $(1, n)$, then $M \setminus B$ can be fibered by annuli parallel to $\gamma_n \times (0,1)$. There are diffeomorphisms of $S^1 \times S^2$ that relate all of these fibrations but the fibrations coming from γ_0 and γ_1 are not isotopic. There are examples of fibrations that are not even diffeomorphic.

DEFINITION 2.3. An *abstract open book* is a pair (Σ, ϕ) where

(1) Σ is an oriented compact surface with boundary and
(2) $\phi : \Sigma \to \Sigma$ is a diffeomorphism such that ϕ is the identity in a neighborhood of $\partial\Sigma$. The map ϕ is called the *monodromy*.

We begin by observing that given an abstract open book (Σ, ϕ) we get a 3-manifold M_ϕ as follows:

$$M_\phi = \Sigma_\phi \cup_\psi \left(\coprod_{|\partial\Sigma|} S^1 \times D^2 \right),$$

where $|\partial\Sigma|$ denotes the number of boundary components of Σ and Σ_ϕ is the mapping torus of ϕ. By this we mean

$$\Sigma \times [0,1]/\sim,$$

where $\sim$ is the equivalence relation $(\phi(x), 0) \sim (x, 1)$ for all $x \in \Sigma$. Finally, $\cup_\psi$ means that the diffeomorphism ψ is used to identify the boundaries of the two manifolds. For each boundary component l of Σ the map $\psi : \partial(S^1 \times D^2) \to l \times S^1 \subset \Sigma_\phi$ is defined to be the unique (up to isotopy) diffeomorphism that takes $S^1 \times \{p\}$ to l where $p \in \partial D^2$ and $\{q\} \times \partial D^2$ to $(\{q'\} \times [0,1]/\sim) = S^1$, where $q \in S^1$ and $q' \in \partial\Sigma$. We denote the cores of the solid tori in the definition of M_ϕ by B_ϕ.

Two abstract open book decompositions (Σ_1, ϕ_1) and (Σ_2, ϕ_2) are called *equivalent* if there is a diffeomrophism $h : \Sigma_1 \to \Sigma_2$ such that $h \circ \phi_1 = \phi_2 \circ h$.

LEMMA 2.4. *We have the following basic facts about open books and abstract open books:*

(1) *An open book decomposition* (B, π) *of* M *gives an abstract open book* (Σ_π, ϕ_π) *such that* $(M_{\phi_\pi}, B_{\phi_\pi})$ *is diffeomorphic to* (M, B).
(2) *An abstract open book determines* M_ϕ *and an open book* (B_ϕ, π_ϕ) *up to diffeomorphism.*
(3) *Equivalent open books give diffeomorphic 3-manifolds.*

EXERCISE 2.5. Prove this lemma.

REMARK 2.6. Clearly the two notions of open book decomposition are closely related. The basic difference is that when discussing open books (non-abstract) we can discuss the binding and pages up to *isotopy* in M, whereas when discussing abstract open books we can only discuss them up to *diffeomorphism*. Thus when discussing Giroux's Theorem 1.1 we need to use (non-abstract) open books; however, it is still quite useful to consider abstract open books and we will frequently not make much of a distinction between them.