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Double Points of Exact Lagrangian Immersions and Legendrian Contact Homology

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Abstract. We use contact homology to obtain lower bounds on the number of double points of self transverse exact Lagrangian immersions of closed manifolds into the product of the cotangent bundle of a manifold and $\mathbb{C}$. The inequality obtained is similar to the Morse inequalities estimating the number of critical points of a Morse function on a closed manifold in terms of its homology.

1. Introduction

Let M be a smooth manifold of dimension n. Consider the cotangent bundle $T^*M \xrightarrow{\pi} M$. The *canonical* 1*-form* θ_M on T^*M maps a tangent vector $X \in T_\alpha(T^*M)$ to $\alpha(d\pi(X))$. The *standard symplectic form* on T^*M is $\omega_M = d\theta_M$. If $(q_1, \dots, q_n)$ are local coordinates on M and $(q_1, \dots, q_n, p_1, \dots, p_n)$ are corresponding coordinates on T^*M then $\theta_M = \sum_j p_j\, dq_j$ and $\omega_M = \sum_j dp_j \wedge dq_j$.

An immersion $f\colon L \to T^*M$ of an n-dimensional manifold L is *Lagrangian* if $f^*\omega_M = 0$. This implies that the form $f^*\theta_M$ is closed. A Lagrangian immersion $f\colon L \to T^*M$ is *exact* if the form $f^*\theta_M$ is exact.

Let $f\colon L \to T^*M$ be an exact Lagrangian immersion of a connected manifold and let $h\colon L \to \mathbb{R}$ be a function such that $dh = f^*\theta_M$. Consider the map $\tilde{f} = (f, h)\colon L \to T^*M \times \mathbb{R} \approx J^1(M)$, where $J^1(M)$ is the 1-jet space of M. This map is an immersion which is everywhere tangent to the hyperplane field $\xi = \ker(dz - \theta_M)$ on $J^1(M)$, where z is a coordinate along the $\mathbb{R}$-direction in $T^*M \times \mathbb{R}$. The hyperplane field ξ is completely non-integrable: if $\alpha = dz - \theta_M$ then $\alpha \wedge (d\alpha)^n \neq 0$. Such a hyperplane field is called a *contact structure* and the 1-form α a *contact form*. In fact ξ is the *standard contact structure* on $J^1(M)$ and α the *standard contact form*. An immersion of an n-manifold into $J^1(M)$ which is everywhere tangent to ξ is called *Legendrian*. Thus, to each exact Lagrangian immersion $f\colon L \to T^*M$ corresponds a family of Legendrian immersions $\tilde{f}\colon L \to J^1(M)$, two members of which differ by a translation in the $\mathbb{R}$-direction (the choice of h is unique up

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to addition of constants). Moreover, for a dense open set of exact Lagrangian immersions their Legendrian lifts are embeddings.

Legendrian and Lagrangian immersions are "soft" in the sense that they obey so called h-principles, see [**14**]. For example, to determine whether or not two Lagrangian (Legendrian) immersion are regularly homotopic through Lagrangian (Legendrian) immersions is a homotopy theoretic question. In contrast to this there are also "hard" properties. For example, double points of exact Lagrangian immersions can in general not be removed even though there are no homotopy obstructions for doing so, as the following theorem of Gromov [**15**] shows.

Theorem 1.1. *An exact Lagrangian immersion* $f\colon L \to \mathbb{C}^n$ *has at least one double point.*

We will present a proof of Gromov's result which uses Floer homology, see Theorem 2.8, and use similar techniques to demonstrate that the following conjecture, see [**1**], (which we state in its simplest form) holds for a certain class of exact Lagrangian submanifolds.

Conjecture 1.2. *Every self transverse Lagrangian immersion* $f\colon L \to \mathbb{C}^n$ *has at least*

$$\frac{1}{2}\dim(H_*(L;\mathbb{Z}_2))$$

double points.

The tool we use is *Legendrian contact homology*, which is part of Symplectic Field Theory, see [**5**] and also [**3**] and [**4**], and is similar to the Floer homology of Lagrangian intersections. It provides Legendrian isotopy invariants via pseudo-holomorphic curve techniques. Using a Morse-Bott argument it is straightforward to show that Conjecture 1.2 holds for any exact Lagrangian the Legendrian lift of which admits a generating function, see e.g. [**2**] or [**6**] for the definition of a generating function. In Theorem 3.5 we prove a result which implies that Conjecture 1.2 holds for exact Lagrangian immersions into $T^*(M \times \mathbb{R})$ provided their Legendrian lifts have good contact homology algebras (see Subsection 3.2 for the definition of a good algebras). This result was first proved in [**9**].

Remark 1.3. The definitions of Floer homology and contact homology given below are streamlined in the sense that only the part of these theories needed for the proof of the double point estimates discussed above will be described. In particular, there is no mention of the grading in either of the theories. Also, for simplicity we use only $\mathbb{Z}_2$-coefficients throughout. If the Legendrian submanifolds considered in Section 3 are assumed to be spin then the $\mathbb{Z}_2$ in all double point estimates involving homology groups could be replaced by $\mathbb{Z}_p$, where p is any prime or with $\mathbb{Q}$, see [**9**].

2. Floer homology and non-injectivity of exact Lagrangian immersions

The purpose of this section is to show that the Floer homology of two compact embedded exact Lagrangian submanifolds of a cotangent bundle T^*M of some n-manifold M is well-defined.

2.1. Floer homology of Lagrangian intersections.

Let L be an embedded exact Lagrangian submanifolds in a cotangent bundle T^*M and let J be an almost complex structure on T^*M compatible with ω_M. That is, ω is positive on J-complex lines and J is an ω-isomorphism. Let S be a Riemann surface with complex structure i. A map $u\colon S \to \mathbb{C}^n$ is called *J-holomorphic* is

$$du + J \circ du \circ i = 0.$$

LEMMA 2.1. *Let S be the unit disk or the Riemann sphere. The only J-holomorphic maps $u\colon S \to T^*M$ such that $u(\partial S) \subset L$ are the constant maps.*

PROOF. Note that the area of a J-holomorphic map $u\colon S \to T^*M$ agrees with its energy and satisfies

$$\operatorname{Area}(u) = \int_S u^*\omega = \int_{\partial S} u^*\theta = \int_{\partial S} dh = 0.$$

Thus any such map must be constant. □

Let L_0 and L_1 be exact Lagrangian transverse submanifolds of T^*M. Let $\mathcal{C} = \{c_1, \dots, c_m\}$ be the set of intersection points of L_1 and L_2. Let $\mathbb{Z}_2\langle\mathcal{C}\rangle$ be the vector space over $\mathbb{Z}_2$ generated by $\mathcal{C}$. We define the Floer homology differential on $\mathbb{Z}_2\langle\mathcal{C}\rangle$ by counting rigid J-holomorphic strips. More precisely, define for double points a and b the moduli space $\mathcal{M}(a;b)$ as the space of maps $u\colon \mathbb{R} \times [0,1] \to T^*M$ such that

- u is J-holomorphic, i.e. $du + J \circ du \circ i = 0$,
- $u(\mathbb{R} \times \{0\}) \subset L_0$ and $u(\mathbb{R} \times \{1\}) \subset L_1$, and
- $\lim_{\tau\to-\infty} u(\tau + it) = a$ and $\lim_{\tau\to\infty} u(\tau + it) = b$,

up to conformal reparametrization. The following lemma is proved in [**11**].

LEMMA 2.2. *For almost complex structures J in an open dense subset $\mathcal{M}(a;b)$ is a finite collection of finite dimensional manifolds with natural compactifications. In particular the 0-dimensional components of the space $\mathcal{M}(a;b)$ form a finite collection of points.*

DEFINITION 2.3. The Floer homology differential $\partial\colon Z_2\langle\mathcal{C}\rangle \to \mathbb{Z}_2\langle\mathcal{C}\rangle$ is the linear map defined on generators as

$$\partial a = \sum_{\dim \mathcal{M}(a;b)=0} |\mathcal{M}(a;b)| b,$$

where $|\mathcal{M}(a;b)|$ denotes the mod 2 number of points in the finite set $\mathcal{M}(a;b)$.

LEMMA 2.4. *The Floer homology differential is a differential, in other words, $\partial^2 = 0$.*

PROOF. To show this one applies the usual gluing argument, see [**12**]. Let a be a double point. A term contributing to $\partial^2 a$ arises through a rigid strip connecting a to b and another rigid strip connecting b to c. These strips can be glued together to a 1-parameter family of strips connecting a to c. Using Gromov compactness we find that this 1-parameter family must break. This can happen in three ways: either the strip splits off a non-constant J-holomorphic sphere or a J-holomorphic disk with boundary on L or it breaks into a rigid strip from a to b' and from b' to c. The two first cases are ruled out by Lemma 2.1. Therefore the contributions to $\partial^2 a$ cancel in pairs and the lemma holds. □

LEMMA 2.5. *The Floer homology $\ker(\partial)/\operatorname{Im}(\partial)$ is invariant under deformations of L_0 and L_1 through exact Lagrangian submanifolds.*