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# Symplectic 4-manifolds, Singular Plane Curves, and Isotopy Problems

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Abstract. We give an overview of various recent results concerning the topology of symplectic 4-manifolds and singular plane curves, using branched covers and isotopy problems as a unifying theme. While this paper does not contain any new results, we hope that it can serve as an introduction to the subject, and will stimulate interest in some of the open questions mentioned in the final section.

## 1. Introduction

An important problem in 4-manifold topology is to understand which manifolds carry symplectic structures (i.e., closed non-degenerate 2-forms), and to develop invariants that can distinguish symplectic manifolds. Additionally, one would like to understand to what extent the category of symplectic manifolds is richer than that of Kähler (or complex projective) manifolds. Similar questions may be asked about singular curves inside, e.g., the complex projective plane. The two types of questions are related to each other via symplectic branched covers.

A branched cover of a symplectic 4-manifold with a (possibly singular) symplectic branch curve carries a natural symplectic structure. Conversely, using approximately holomorphic techniques it can be shown that every compact symplectic 4-manifold is a branched cover of the complex projective plane, with a branch curve presenting nodes (of both orientations) and complex cusps as its only singularities (cf. §3). The topology of the 4-manifold and that of the branch curve are closely related to each other; for example, using braid monodromy techniques to study the branch curve, one can reduce the classification of symplectic 4-manifolds to a (hard) question about factorizations in the braid group (cf. §4). Conversely, in some examples the topology of the branch curve complement (in particular its fundamental group) admits a simple description in terms of the total space of the covering (cf. §5).

In the language of branch curves, the failure of most symplectic manifolds to admit integrable complex structures translates into the failure of most symplectic branch curves to be isotopic to complex curves. While the symplectic isotopy

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problem has a negative answer for plane curves with cusp and node singularities, it is interesting to investigate this failure more precisely. Various partial results have been obtained recently about situations where isotopy holds (for smooth curves; for curves of low degree), and about isotopy up to stabilization or regular homotopy (cf. §6). On the other hand, many known examples of non-isotopic curves can be understood in terms of twisting along Lagrangian annuli (or equivalently, Luttinger surgery of the branched covers), leading to some intriguing open questions about the topology of symplectic 4-manifolds versus that of Kähler surfaces.

## 2. Background

In this section we review various classical facts about symplectic manifolds; the reader unfamiliar with the subject is referred to the book [**19**] for a systematic treatment of the material.

Recall that a *symplectic form* on a smooth manifold is a 2-form $\omega$ such that $d\omega = 0$ and $\omega \wedge \cdots \wedge \omega$ is a volume form. The prototype of a symplectic form is the 2-form $\omega_0 = \sum dx_i \wedge dy_i$ on $\mathbb{R}^{2n}$. In fact, one of the most classical results in symplectic topology, Darboux's theorem, asserts that every symplectic manifold is locally symplectomorphic to $(\mathbb{R}^{2n}, \omega_0)$: hence, unlike Riemannian metrics, symplectic structures have no local invariants.

Since we are interested primarily in compact examples, let us mention compact oriented surfaces (taking $\omega$ to be an arbitrary area form), and the complex projective space $\mathbb{CP}^n$ (equipped with the Fubini-Study Kähler form). More generally, since any submanifold to which $\omega$ restricts non-degenerately inherits a symplectic structure, all complex projective manifolds are symplectic. However, the symplectic category is strictly larger than the complex projective category, as first evidenced by Thurston in 1976 [**36**]. In 1994 Gompf obtained the following spectacular result using the *symplectic sum* construction [**14**]:

Theorem 1 (Gompf). *Given any finitely presented group $G$, there exists a compact symplectic 4-manifold $(X, \omega)$ such that $\pi_1(X) \simeq G$.*

Hence, a general symplectic manifold cannot be expected to carry a complex structure; however, we can equip it with a compatible *almost-complex* structure, i.e. there exists $J \in \mathrm{End}(TX)$ such that $J^2 = -\mathrm{Id}$ and $g(\cdot,\cdot) := \omega(\cdot, J\cdot)$ is a Riemannian metric. Hence, at any given point $x \in X$ the tangent space $(T_xX, \omega, J)$ can be identified with $(\mathbb{C}^n, \omega_0, i)$, but there is no control over the manner in which $J$ varies from one point to another ($J$ is not *integrable*). In particular, the $\bar\partial$ operator associated to $J$ does not satisfy $\bar\partial^2 = 0$, and hence there are no local holomorphic coordinates.

An important problem in 4-manifold topology is to understand the hierarchy formed by the three main classes of compact oriented 4-manifolds: (1) complex projective, (2) symplectic, and (3) smooth. Each class is a proper subset of the next one, and many obstructions and examples are known, but we are still very far from understanding what exactly causes a smooth 4-manifold to admit a symplectic structure, or a symplectic 4-manifold to admit an integrable complex structure.

One of the main motivations to study symplectic 4-manifolds is that they retain some (but not all) features of complex projective manifolds: for example the structure of their Seiberg-Witten invariants, which in both cases are non-zero and count certain embedded curves [**31, 32**]. At the same time, every compact oriented smooth 4-manifold with $b_2^+ \geq 1$ admits a "near-symplectic" structure, i.e. a closed 2-form which vanishes along a union of circles and is symplectic over the complement of its zero set [**13, 16**]; and it appears that some structural properties of symplectic manifolds carry over to the world of smooth 4-manifolds (see e.g. [**33, 5**]).

Many new developments have contributed to improve our understanding of symplectic 4-manifolds over the past ten years (while results are much scarcer in higher dimensions). Perhaps the most important source of new results has been the study of pseudo-holomorphic curves in their various incarnations: Gromov-Witten invariants, Floer homology, ... (for an overview of the subject see [**20**]). At the same time, gauge theory (mostly Seiberg-Witten theory, but also more recently Ozsváth-Szabó theory) has made it possible to identify various *obstructions* to the existence of symplectic structures in dimension 4 (cf. e.g. [**31, 32**]). On the other hand, various new constructions, such as link surgery [**11**], symplectic sum [**14**], and symplectic rational blowdown [**30**] have made it possible to exhibit interesting families of non-Kähler symplectic 4-manifolds. In a slightly different direction, approximately holomorphic geometry (first introduced by Donaldson in [**9**]) has made it possible to obtain various structure results, showing that symplectic 4-manifolds can be realized as symplectic Lefschetz pencils [**10**] or as branched covers of $\mathbb{CP}^2$ [**2**]. In the rest of this paper we will focus on this latter approach, and discuss the topology of *symplectic branched covers* in dimension 4.

## 3. Symplectic branched covers

Let $X$ and $Y$ be compact oriented 4-manifolds, and assume that $Y$ carries a symplectic form $\omega_Y$.

Definition 2. *A smooth map $f : X \to Y$ is a* symplectic branched covering *if given any point $p \in X$ there exist neighborhoods $U \ni p$, $V \ni f(p)$, and local coordinate charts $\phi : U \to \mathbb{C}^2$ (orientation-preserving) and $\psi : V \to \mathbb{C}^2$ (adapted to $\omega_Y$, i.e. such that $\omega_Y$ restricts positively to any complex line in $\mathbb{C}^2$), in which $f$ is given by one of:*

*(i) $(x, y) \mapsto (x, y)$ (local diffeomorphism),*
*(ii) $(x, y) \mapsto (x^2, y)$ (simple branching),*
*(iii) $(x, y) \mapsto (x^3 - xy, y)$ (ordinary cusp).*

These local models are the same as for the singularities of a generic holomorphic map from $\mathbb{C}^2$ to itself, except that the requirements on the local coordinate charts have been substantially weakened. The *ramification curve* $R = \{p \in X,\ \det(df) = 0\}$ is a smooth submanifold of $X$, and its image $D = f(R)$ is the *branch curve*, described in the local models by the equations $z_1 = 0$ for $(x, y) \mapsto (x^2, y)$ and $27z_1^2 = 4z_2^3$ for $(x, y) \mapsto (x^3 - xy, y)$. The conditions imposed on the local coordinate charts imply that $D$ is a *symplectic curve* in $Y$ (i.e., $\omega_{Y|TD} > 0$ at every point of $D$). Moreover the restriction of $f$ to $R$ is an immersion everywhere except at the cusps. Hence, besides the ordinary complex cusps imposed by the local model, the only generic singularities of $D$ are transverse double points ("nodes"), which may occur with either the complex orientation or the anti-complex orientation.

We have the following result [**2**]: