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The André-Oort conjecture for products of modular curves

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Abstract. In this paper we prove (assuming the Generalised Riemann Hypothesis) the André-Oort conjecture for products of modular curves using a combination of Galois-theoretic and ergodic-theoretic methods.

1. Introduction

The André-Oort conjecture stated below has been recently proved by Klingler, Ullmo and Yafaev (see [**UY06**] and [**KY06**]) in full generality assuming the Generalised Riemann Hypothesis.

Conjecture 1.1 (André-Oort). *Let S be a Shimura variety and let Σ be a set of special points in S. Every irreducible component of the Zariski closure of Σ is a special (or Hodge type) subvariety of S.*

For generalities on this conjecture, in particular for the notions of special points and subvarieties we refer, for example, to [**Yaf07**]. The purpose of this note is to present a proof of this conjecture in the special case where S is a product of an arbitrary number of modular curves. It is our hope that this will help in understanding the strategy used in [**UY06**] and [**KY06**], as many of the technical problems occurring in the general case do not present themselves in the case considered in this paper but all of the main ideas of the proof are conserved. The main result of this paper is the following.

Theorem 1.2. *Assume the GRH for imaginary quadratic fields. Let $n \geq 1$ be an integer and let S be a product of n modular curves. Let Σ be a set of special points in S. The irreducible components of the Zariski closure of Σ are special subvarieties.*

Note that this case of the conjecture has already been dealt with by Edixhoven [**Edi05**] but his strategy does not seem to be easily generalisable as it relies on the very particular geometric properties of the Shimura variety under consideration. We also point out that our strategy yields a proof of the Manin-Mumford conjecture as well (the "abelian counterpart" of the André-Oort conjecture). We refer to [**RU**] for details on this.

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The strategy of the proof is based on the following alternative in the geometry of Shimura varieties. Let S be a Shimura variety and let Z_n be a sequence of irreducible special subvarieties of S. Let F be some number field over which S admits a canonical model. After possibly replacing Z_n by a subsequence and assuming the GRH for CM-fields, at least one of the following cases occurs.

(1) The cardinality of the sets $\{\sigma(Z_n), \sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/F)\}$ is unbounded as $n \to \infty$ (and therefore Galois-theoretic techniques can be used).
(2) The sequence of probability measures μ_n canonically associated to Z_n weakly converges to some μ_Z, the probability measure canonically associated to a special subvariety Z of S. Moreover, for every n large enough, Z_n is contained in Z.

Which of the two cases occurs depends on the geometric nature of the subvarieties Z_n.

Let us explain this in more detail in the case considered in this paper. So let S be a product of n modular curves. We assume that S is $(\mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H})^n = \mathbb{C}^n$. Special subvarieties are products of factors which are of one of the following forms:

(1) A special point (equivalently CM point) of some $\mathbb{C}^m$, $m \leq n$.
(2) A modular curve $\Gamma\backslash\mathbb{H}$ (for some congruence subgroup of Γ of $\mathrm{SL}_2(\mathbb{Z})$) embedded in a product of copies of $\mathbb{C}$.

A special subvariety is called *strongly* special if it does not have any CM factors. Sequences of strongly special subvarieties are precisely those for which the second case of the alternative occurs (this is a consequence of a theorem of Clozel-Ullmo that we will recall later). The sequences of special subvarieties that do have special factors are those for which the first case of the alternative occurs.

The strategy of the proof is as follows. For a special subvariety Z, we let $c(\Omega_Z)$ be the number of CM factors, therefore $c(\Omega_Z) = 0$ means precisely that Z is strongly special. Let X be a subvariety of S containing a Zariski dense set Σ of special subvarieties. We can assume (after possibly replacing Σ by a Zariski dense subset) that $c(\Omega_Z)$ is constant as Z ranges through Σ; let's call $c(\Sigma)$ this number. If $c(\Sigma) = 0$, then X is special by the theorem of Clozel and Ullmo, otherwise the size of the Galois orbit of Z is unbounded as Z ranges through Σ. Using the explicit description of the Galois action on special points and a characterisation of special subvarieties in terms of Hecke correspondences, we show that every Z with sufficiently large Galois orbit is contained in a special subvariety Z' with $c(\Omega_{Z'}) < c(\Omega_Z)$. Thus we construct a Zariski-dense set Σ' of special subvarieties with $c(\Sigma') < c(\Sigma)$. We then reiterate the process with Σ' instead of Σ. Eventually we obtain a Zariski dense set of strongly special subvarieties.

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2. Preliminaries.

Before we state and prove our main result, we recall some definitions and prove some preliminary results which will be used in the course of the proof. Let us first recall the following definition from Edixhoven ([**Edi05**], definition 1.1).

Definition 2.1. *Let I be a finite set of cardinality r. For every i in I, let Γ_i be a congruence subgroup of $\mathrm{SL}_2(\mathbb{Z})$ and S be the product of the $X_{\Gamma_i} := \Gamma_i\backslash\mathbb{H}$ for $i \in I$. A closed irreducible subvariety Z of S is called special (of type $\Omega = \Omega_Z$) if I has a partition $\Omega = (I_1, \dots, I_t)$ such that Z is a product of subvarieties Z_i of $S_i = \prod_{j\in I_i} \Gamma_j\backslash\mathbb{H}$, each of one of the forms:*

(1) *I_i is a one-element set and Z_i is a CM point.*
(2) *Z_i is the image of $\mathbb{H}$ in S_i under the map sending τ in $\mathbb{H}$ to the image of $(g_s\tau)_{s\in I_i}$ in S_i, where the g_s are some elements of $\mathrm{GL}_2(\mathbb{Q})$ with positive determinant.*

Given a special subvariety Z of type Ω, we define $c(\Omega)$ to be the number of CM factors. A special subvariety Z is called strongly special if $c(\Omega) = 0$

We now prove a few lemmas that will be used in the course of the proof.

Lemma 2.2. *Let Z be a strongly special subvariety of $\mathbb{C}^n$. Then Z is defined (as an absolutely irreducible subscheme) over an abelian extension L of $\mathbb{Q}$ such that $\mathrm{Gal}(L/\mathbb{Q})$ is killed by the multiplication by 2, i.e. for every σ in $\mathrm{Gal}(L/\mathbb{Q})$, $\sigma^2 = 1$.*

Proof. This is a consequence of the explicit description of the Galois action on irreducible components of strongly special subvarieties via a reciprocity law. We refer to section 2 of [**UY06**] for details on this.

The inclusion $Z \hookrightarrow \mathbb{C}^n$ corresponds to the inclusion of Shimura data

$$(\mathrm{PGL}_2^m, \mathbb{H}^{\pm m}) \hookrightarrow (\mathrm{PGL}_2^n, \mathbb{H}^{\pm n})$$

for some $m \leq n$. This is a consequence of the explicit description of special subvarieties of $\mathbb{C}^n$ given above. Let $\rho\colon \mathrm{SL}_2^m \longrightarrow \mathrm{PGL}_2^m$ be the simply connected covering. Its kernel is killed by 2. Then the reciprocity morphism defining the Galois action on connected components is a morphism

$$r\colon \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \longrightarrow \mathrm{PGL}_2^m(\mathbb{A}_f)/\mathrm{PGL}_2^m(\mathbb{Q})\rho(\mathrm{SL}_2^m(\mathbb{A}_f))$$

It is now clear that its image (which is isomorphic to $\mathrm{Gal}(L/\mathbb{Q})$) is killed by 2. □

Using the above lemma we now prove the following.

Lemma 2.3. *Let $Z = \{x_1, \dots, x_s\} \times Z'$ be a special subvariety of $\mathbb{C}^n$ (Z' is a strongly special subvariety of $\mathbb{C}^{n-s}$ and $(x_1, \dots, x_s)$ is a special point of $\mathbb{C}^s$). Let O_{x_i} be the ring of complex multiplication of the point x_i. Let l be a prime splitting every O_{x_i}. Let $T_{\mathbf{l^2}}$ be the Hecke correspondence defined by the element of the product of r copies of $\mathrm{GL}_2(\mathbb{Q})^+$ which is $\left(\begin{smallmatrix} l^2 & 0 \\ 0 & 1\end{smallmatrix}\right)$ on the first s components and 1 elsewhere.*

There exists an element σ of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ such that

$$\sigma(Z) \subset T_{\mathbf{l^2}} Z$$

Proof. Let K be the composite of the fields K_{x_i} of complex multiplication of the points x_i and let R be the ring $O_{x_1} \otimes \cdots \otimes O_{x_s}$. The ring R is an order in K and the prime l splits in R. Let τ be the Frobenius element in $\mathrm{Gal}(\overline{\mathbb{Q}}/K)$ for a prime ideal lying over l. The theory of complex multiplication of elliptic curves shows that $\tau^2(x_i) \subset T_{l^2}(x_i)$ where T_{l^2} is the usual Hecke correspondence given by the element