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Moduli of abelian varieties and p-divisible groups

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Abstract. This is a set of notes for a course we gave in the second week of August in the 2006 CMI Summer School at Göttingen. Our main topic is geometry and arithmetic of $\mathcal{A}_g \otimes \mathbb{F}_p$, the moduli space of polarized abelian varieties of dimension g in positive characteristic. We illustrate properties of $\mathcal{A}_g \otimes \mathbb{F}_p$, and some of the available techniques by treating two topics: 'Density of ordinary Hecke orbits' and 'A conjecture by Grothendieck on deformations of p-divisible groups'.

Contents

We present proofs of two recent results. The main point is that the *methods* used for these proofs are interesting. The emphasis will be on the various techniques available.

In characteristic zero we have strong tools at our disposal: besides algebraic-geometric theories we can use analytic and topological methods. It would seem that we are at a loss in positive characteristic. However the opposite is true. Phenomena

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occurring only in positive characteristic provide us with strong tools to study moduli spaces. And, as it turns out again and again, several results in characteristic zero can be derived using reduction modulo p. The discussion of *tools in positive characteristic* will be the focus of our notes.

Here is a list of some of the central topics:

- Serre-Tate theory.
- Abelian varieties over finite fields.
- Monodromy: ℓ-adic and p-adic, geometric and arithmetic.
- Dieudonné modules and Newton polygons.
- Theory of Dieudonné modules, Cartier modules and displays.
- Cayley-Hamilton and deformations of p-divisible groups.
- Hilbert modular varieties.
- Purity of the Newton polygon stratification in families of p-divisible groups.

The strategy is that we have chosen certain central topics, and for those we took ample time for explanation and for proofs. Besides that we need certain results which we label as "Black Box". These are results which we need for our proofs, which are either fundamental theoretical results (but it would take too much time to explain their proofs), or lemmas which are computational, important for the proof, but not very interesting to explain in a course. We hope that we explain well enough what every relevant statement is. We write:

BB A Black Box, please accept that this result is true.

Th One of the central results, we will explain it.

Extra A result which is interesting but was not discussed in the course.

Notation to be used will be explained in Section 10. In order to be somewhat complete we will gather related interesting results, questions and conjectures in Section 11. Part of our general convention is that K denotes a field of characteristic $p > 0$, unless otherwise specified, and k denotes an algebraically closed field.

We assume that the reader is familiar with the basic theory of abelian varieties at the level of Chapter II of [**54**] and [**55**], Chapter 6; we consider abelian varieties over an arbitrary field, and abelian schemes over a base scheme. Alternative references: [**16**], [**81**]. For the main characters of our play: *abelian varieties, moduli spaces, and p-divisible groups*, we give references and definitions in Section 10.

1. Introduction: Hecke orbits, and the Grothendieck conjecture

In this section we discuss the two theorems we are going to consider.

1.1. An abelian variety A of dimension g over a field $K \supset \mathbb{F}_p$ is said to be *ordinary* if

$$\#\left(A[p](k)\right) = p^g.$$

More generally, the number $f = f(A)$ such that $\#\left(A[p](k)\right) = p^f$ is called the *p-rank* of A. It is a fact that the p-rank of A is at most $\dim(A)$; an abelian variety is ordinary if its p-rank is equal to its dimension. See 10.10 for other equivalent definitions.

An elliptic curve E over a field $K \supset \mathbb{F}_p$ is said to be *supersingular* if it is not ordinary; equivalently, E is supersingular if $E[p](k) = 0$ for any overfield $k \supset K$. This terminology stems from Deuring: an elliptic curve in characteristic zero is said to determine a *singular* j-value if its endomorphism ring over an algebraically closed field (of characteristic 0) is larger than $\mathbb{Z}$ (therefore of rank 2 over $\mathbb{Z}$), while a *supersingular* elliptic curve E over an algebraically closed field $k \supset \mathbb{F}_p$ has $\mathrm{rk}_{\mathbb{Z}}(\mathrm{End}(E)) = 4$. Since an elliptic curve is non-singular, a better terminology would be "an elliptic curve with a singular j-invariant".

We say an abelian variety A of dimension g over a field K is *supersingular* if there exists an isogeny $A \otimes_K k \sim E^g$, where E is a supersingular elliptic curve. An equivalent definition for an abelian variety in characteristic p to be supersingular is that all of its slopes are equal to $1/2$; see 4.38 for the definition of slopes and the Newton polygon. Supersingular abelian varieties have p-rank zero. For $g = 2$ one can show that (supersingular) $\Leftrightarrow$ $(f = 0)$, where f is the p-rank. For $g > 2$ there exist abelian varieties of p-rank zero which are not supersingular, see 5.22.

Hecke orbits

Definition 1.2. Let A and B be abelian varieties over a field K. Let $\Gamma \subset \mathbb{Q}$ be a subring. A Γ-isogeny from A to B is an element ψ of $\mathrm{Hom}(A, B) \otimes_{\mathbb{Z}} \Gamma$ which has an inverse in $\mathrm{Hom}(B, A) \otimes_{\mathbb{Z}} \Gamma$, i.e., there exists an element $\psi' \in \mathrm{Hom}(B, A) \otimes_{\mathbb{Z}} \Gamma$ such that $\psi' \psi = \mathrm{id}_A \otimes 1$ in $\mathrm{Hom}(A, A) \otimes_{\mathbb{Z}} \Gamma$ and $\psi \psi' = \mathrm{id}_B \otimes 1$ in $\mathrm{Hom}(B, B) \otimes_{\mathbb{Z}} \Gamma$.

Remark.

(i) When $\Gamma = \mathbb{Q}$ (resp. $\Gamma = \mathbb{Z}_{(p)}$, resp. $\mathbb{Z}[1/\ell]$), we say that ψ is a *quasi-isogeny* (resp. *prime-to-p quasi-isogeny*, resp. an *ℓ-power quasi-isogeny*). A *prime-to-p isogeny* (resp. *ℓ-power isogeny*) is an isogeny which is also a $\mathbb{Z}_{(p)}$-isogeny (resp. a $\mathbb{Z}[1/\ell]$-isogeny). Here $\mathbb{Z}_{(p)} = \mathbb{Q} \cap \mathbb{Z}_p$ is the localization of $\mathbb{Z}$ at the prime ideal $(p) = p\mathbb{Z}$.

(ii) A $\mathbb{Q}$-isogeny ψ (resp. $\mathbb{Z}_{(p)}$-isogeny, resp. $\mathbb{Z}[1/\ell]$-isogeny) can be realized by a diagram

$$A \xleftarrow{\alpha} C \xrightarrow{\beta} B\,,$$

where α and β are isogenies such that there exists an integer $N \in \Gamma^{\times}$ (resp. an integer N prime to p, resp. an integer $N \in \ell^{\mathbb{N}}$) such that $N \cdot \mathrm{Ker}(\alpha) = N \cdot \mathrm{Ker}(\beta) = 0$.

Definition 1.3. Let $[(A, \lambda)] = x \in \mathcal{A}_g(K)$ be the moduli point of a polarized abelian variety over a field K.

(i) We say that a point $[(B, \mu)] = y$ of $\mathcal{A}_g$ is in the *Hecke orbit* of x if there exists a field Ω and

$$\text{a } \mathbb{Q}\text{-isogeny } \varphi : A_\Omega \to B_\Omega \text{ such that } \varphi^*(\mu) = \lambda.$$

Notation: $y \in \mathcal{H}(x)$. The set $\mathcal{H}(x)$ is called the *Hecke orbit* of x.

(ii) *Hecke-prime-to-p-orbits.* If in the previous definition moreover φ is a $\mathbb{Z}_{(p)}$-isogeny, we say $[(B, \mu)] = y$ is in the Hecke-prime-to-p-orbit of x.
Notation: $y \in \mathcal{H}^{(p)}(x)$.

(iii) *Hecke-ℓ-orbits.* Fix a prime number ℓ different from p. We say $[(B, \mu)] = y$ is in the Hecke-ℓ-orbit of x if in the previous definition moreover φ is a $\mathbb{Z}[1/\ell]$-isogeny.
Notation: $y \in \mathcal{H}_\ell(x)$.